

Is there a universal log law for turbulent wall-bounded flows?

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The history and theory supporting the idea of a universal log law for turbulent wall-bounded flows are briefly reviewed. The original idea of justifying a log law from a constant Reynolds stress layer argument is found to be deficient. By contrast, it is argued that the logarithmic friction law and velocity profiles derived from matching inner and outer profiles for a pipe or channel flow are well-founded and consistent with the data. But for a boundary layer developing along a flat plate it is not, and in fact it is a power law theory that seems logically consistent. Even so, there is evidence for at least an empirical logarithmic fit to the boundary-friction data, which is indistinguishable from the power law solution. The value of $\kappa \approx 0.38$ obtained from a logarithmic curve fit to the boundary-layer velocity data, however, does not appear to be the same as for pipe flow for which 0.43 appears to be the best estimate. Thus, the idea of a universal log law for wall-bounded flows is not supported by either the theory or the data.

Keywords: wall turbulence; log law; pipe; channel; boundary layer

1. Introduction

This paper examines the state of our knowledge about the log law in wall-bounded turbulent flows, whether it exists at all, and whether it should be expected to be universal. Several simple flows will be considered: turbulent pipe and channel flows, and the turbulent boundary layer that develops along a flat plate with zero pressure gradient and is two-dimensional in the mean. Attention will be focused entirely on smooth walls for which surface imperfections have no measurable influence.

All of the flows we consider herein will be assumed to have undergone transition and to be describable as fully developed. For pipe and channel flows, this means that all the statistics have reached a state in which the only non-zero streamwise derivative is the imposed mean pressure gradient, which itself is a constant. The boundary layer never achieves this state and continues to evolve forever. Hence, fully developed in this case means that whatever remains from the laminar to turbulent transition at most affects that large scale structures in a way that is replicated over and over. Note that this does not assume that developing turbulent shear flows become asymptotically independent of their upstream conditions.

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The role of the Reynolds number is also different for the two types of flows. For fully developed pipe and channel flows, the Reynolds numbers characterizing the flow are fixed by the imposed conditions (e.g. streamwise pressure gradient, channel height or pipe diameter, and the mass flow rate, only two of which can be independent). This means that the relative effects of viscous to inertial terms are independent of streamwise distance, and their relative balance can vary only across the flow. By contrast, for the boundary layers, the local Reynolds numbers, however defined, evolve continuously, increasing (for most) as the boundary layers grow. Whether it is necessary to account theoretically for this slow variation of the Reynolds numbers (together with the streamwise inhomogeneity causing it) represents the fundamental point of disagreement between the various approaches to turbulent boundary layers, and is at the heart of most of the debate concerning the validity (or lack thereof) of the log law for developing boundary-layer flows (cf. George & Castillo 1997; George 2006).

2. What is the ‘log law’ and why is it important?

The so-called ‘log law’ is widely believed to describe most (if not all) turbulent wall-bounded flows, and lies at the core of the most widely used engineering computational models involving turbulent flow near surfaces. While there are several manifestations of the log law, the most important is the mean velocity profile normalized in so-called ‘inner’ variables given by

$$U^+ = \frac{1}{\kappa} \ln y^+ + B_i, \quad (2.1)$$

where $U^+ = U/u_*$ and $y^+ = yu_*/\nu$. U is the mean velocity, y is the distance from the wall, ν is the kinematic viscosity and u_* is the friction velocity defined from the wall skin friction, τ_w , as $u_*^2 = \tau_w/\rho$ where ρ is the fluid density.

The empirical constants in the log velocity profile, the so-called von Kármán constant, κ , and the additive constant, B_i , are believed by many to be universal (see almost any text including a chapter on turbulence). Nonetheless, there seems to be little consensus about what these universal values actually are. For several decades following the work of Coles (1968) the value of κ was assumed to be approximately 0.41, but recent (and hotly disputed) estimates place it as low as 0.38 (Nagib *et al.* 2004) and as high as 0.45 (Zagorala & Smits 1998). Some even believe that κ should be $1/e$ (e.g. Zanoun 2003), as much on the desire to attribute to κ some special status as on theoretical argument or hard empirical evidence. There is even less consensus about the value of B_i . Traditional estimates were between 4.9 and 5.1, but recent estimates cover a much wider range from 4 to 10. Surprisingly, the lack of apparent universality of B_i seems to be of far less concern than the precise value of κ . As should be clear from equation (2.1), however, a slight change in one parameter can cause a corresponding change in the other, even for a given dataset.

The differences in κ may appear slight, especially given the errors involved in most turbulence measurements, but in fact they are quite important. The reason for this is that most turbulence models used by the aircraft industry depend in one form or another on the assumption that the flow very close to the wall can be described by the logarithmic velocity profile of equation (2.1). In fact, according to Spalart (2006), a 2% decrease in κ makes a 1% decrease in the overall drag

estimate for a modern aircraft (like the Boeing 787 or the Airbus 350). Obviously, an aircraft company would prefer the lower value in marketing a plane that has not yet been built, since the fuel economy would appear to be much better than a competitor's estimate using a higher value. But this could be a great risk, since eventually the plane has to fly and the real drag will eventually be determined. It is perhaps only coincidence that there are many planes parked in the desert which could not make their design range.

The reason for the sensitivity of the overall drag to the value of κ can be most easily seen from the logarithmic friction law, which for a turbulent boundary layer is usually expressed as

$$c_f = 2 \left[\frac{1}{\kappa} \ln R_\theta + C \right]^{-1} = \frac{2\kappa^2}{[\ln R_\theta + \kappa C]^2}. \quad (2.2)$$

The skin-friction coefficient, c_f , is defined using the wall shear stress, τ_w , and the free stream velocity (measured relative to the surface), U_0 , as

$$c_f = \frac{\tau_w}{[(1/2)\rho U_0^2]}. \quad (2.3)$$

$R_\theta = U_0\theta/\nu$ is the Reynolds number based on the local momentum thickness defined by

$$\theta = \int_0^\infty (U/U_0)[1 - (U/U_0)]dy, \quad (2.4)$$

where y is the direction normal to the surface (so the integration is across the boundary layer). The most recent estimates of Nagib *et al.* (2004) place the value of C at about 4.1 if $\kappa=0.38$. In a typical aerodynamic boundary layer, R_θ increases along the surface, so the skin-friction drag can be obtained by integrating c_f along the surface. Equation (2.2) applies approximately from the region at which the flow becomes turbulent, typically at values of R_θ of a thousand or less. Upstream, the boundary layer is either laminar or transitional, both of which are important but of no concern in this paper.

3. Why is this so difficult?

Typical values of R_θ for boundary-layer experiments in wind tunnels where the near-wall region is resolved in the measurements have only recently achieved 20 000–30 000. These can be compared to the values in flight of several hundred thousand or more. Values of c_f in the laboratory are about 0.002 or greater, but often much less in flight. Simply put, the streamwise extent of a fuselage and the speed of the airplane lead to Reynolds numbers much larger than we can measure in our laboratories, at least at the moment.

The reason that the large discrepancy between experiments and application matters is that the dynamics of the boundary layer change rather dramatically with increasing Reynolds number, at least until some threshold is surpassed. The log region (by the conventional reasoning) can exist only as long as neither viscosity nor mean convection (advection) play a significant role in its dynamics. This means that we should at most expect the mean velocity to be approximately logarithmic between about 30 viscous units and about 10% of the boundary-layer thickness, the distance at which the mean velocity achieves 99% of its free stream

value (i.e. $30 < y^+ < 0.1\delta_{99}^+$). Therefore, $\delta^+ > 300$ (which corresponds to $R_\theta > 1000$ approximately) is needed to observe even the beginnings of a log layer, and a factor of 10 higher is needed to observe a decade (in space) of logarithmic behaviour. There are strong arguments from the dynamics of the turbulent fluctuations (cf. George & Castillo 1997; Wosnik *et al.* 2000) that the true ‘inertial layer’ can actually exist only between approximately $300 < y^+ < 0.1\delta_{99}^+$, and the region between 30 and 300 approximately is a ‘mesolayer’ in which viscosity affects all the turbulence scales of motion, even though it is negligible in the mean momentum equation. This means that $\delta^+ > 3000$ (corresponding to $R_\theta > 10\,000$) is a necessary condition for a true inertially dominated log layer to even exist. And R_θ must be much higher if a substantial region of logarithmic behaviour is to be observed, hence our failure to make much sense out of the experiments at lower Reynolds numbers.

So why have not we carried out experiments at Reynolds numbers more representative of engineering needs? Most techniques for accurate measurement of turbulence require incompressible flows at quite low speeds (typically 10–20 m s⁻¹). The reasons have both to do with the problems presented by compressibility effects on the measuring technique and the size of the smallest probes we can build. For the latter, the scale of the disturbances being resolved must be larger than the largest dimension of the probe or measuring volume. For boundary layers, the smallest scale of interest is the viscous length-scale (or wall length-scale) defined as $\eta_w = \nu/u_*$. Clearly, the larger the velocity, the smaller the value of η_w (assuming ν to be constant). The smallest practical hot-wire probes are typically 0.2 μm in diameter and larger than 0.5 mm in length, but they have significant problems operating within a few tens of diameters from the surface due to enhanced heat transfer to the surface. The resolution of modern optical techniques is about the same, although micro-PIV (particle image velocimetry) and micro-LDA have been able to measure to within 10 μm of the wall.

The largest boundary-layer facilities in the world at the moment with sufficient flow quality for this type of research are at the Laboratoire Mécanique de Lille (LML) in France and at the University of Melbourne in Australia. Both wind tunnels have a test section approximately 20 m in length. This length allows the boundary layer at 10 m s⁻¹ to grow to approximately 0.3 m thickness, which in turn allows the flow to be measured to within one viscous length from the wall, $y = \nu/u_* = 10 \mu\text{m}$. But even these extraordinary facilities can only accomplish this at a value of $R_\theta = 20\,000$ – $30\,000$, meaning that less than half a decade of an inertially dominated log layer should be expected. Most importantly, this is still well below the Reynolds numbers at which airplanes must fly. Experiments reported at higher Reynolds numbers obviously must settle for much less resolution. As a consequence, we are left with the problem of whether the observed Reynolds number dependencies are really from the flow, or from simply the changing probe response as the flow around it changes with flow speed. The obvious solution (especially since all other alternatives have been exhausted over the past 80 years) is to build much larger facilities in which high Reynolds numbers can be achieved at low speed. Unfortunately, funding agencies (and especially industry) have shown little interest. This may change with aerodynamic design improving to the point where skin friction is more than 50% of the drag in flight, but most likely it will take a colossal failure in drag prediction to get their attention.

4. The pipe/channel flow alternative

Owing to the problems in obtaining boundary-layer data at high Reynolds number, for most of the past century investigators have turned to turbulent pipe and channel flows, arguing that the near-wall behaviour should be the same. The argument goes like this: the region closest to the wall (say inside of 10% of the channel height, pipe radius or boundary-layer thickness) is governed by pretty much the same equations, at least if the Reynolds number is high enough, i.e.

$$0 = \frac{\partial}{\partial y} \left[-\langle uv \rangle + \nu \frac{\partial U}{\partial y} \right]. \quad (4.1)$$

This can be integrated from the wall to obtain

$$-\langle uv \rangle + \nu \frac{\partial U}{\partial y} = \frac{\tau_w}{\rho} = u_*^2, \quad (4.2)$$

which explains why it is often referred to as the ‘constant (total) stress layer’. It was these arguments which lead [von Kármán \(1930\)](#) and [Prandtl \(1932\)](#) to postulate the log law to begin with using a simple eddy viscosity argument. Simply model the Reynolds shear stress with an eddy viscosity, say $-\langle uv \rangle = \nu_e \partial U / \partial y \approx u_*^2$ (since the viscous term is negligible in the inertial region). Then on dimensional grounds choose $\nu_e = \kappa u_* y$, where κ is a constant of proportionality. Integration yields equation (2.1) immediately. Outside of $y^+ = 30$ (but within the 10% outside limit), the results looked pretty good, especially given the data at the time. Hence, not only was the idea of the log law born, but it also came (on the basis of quite limited evidence) to be considered ‘universal’. The subsequent refinements by [Isakson \(1937\)](#), [Millikan \(1938\)](#), [Clauser \(1954\)](#) and [Coles \(1956\)](#) were so strongly embraced by the community that they not only appear in virtually all texts, but it is only in the past decade or so that serious challenges to the arguments are even publishable.

Now, in fact the averaged equations for a turbulent boundary layer (including pressure gradient) look like this

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{\partial P_\infty}{\partial x} + \frac{\partial}{\partial y} \left[-\langle uv \rangle + \nu \frac{\partial U}{\partial y} \right], \quad (4.3)$$

where the y -momentum equation for the boundary layer has been integrated to replace the local pressure by that imposed from outside the boundary layer, P_∞ , which is in turn assumed to be a function of the streamwise coordinate, x , only. By contrast, fully developed turbulent channel flows are homogeneous in the streamwise direction (only the pressure varies with x), so the convection terms (left-hand side) are identically zero and the mean momentum equation reduces to

$$0 = -\frac{1}{\rho} \frac{dP}{dx} + \frac{d}{dy} \left[-\langle uv \rangle + \nu \frac{dU}{dy} \right]. \quad (4.4)$$

The cross-stream momentum equation can be used to argue that P is independent of y , to at least second order in the turbulence intensity.

Clearly, equations (4.3) and (4.4) reduce to equation (4.1) only if the extra terms in each vanish; i.e. the mean convection terms on the left-hand side of equation (4.3) and the pressure gradient term in both. This is *presumed* to

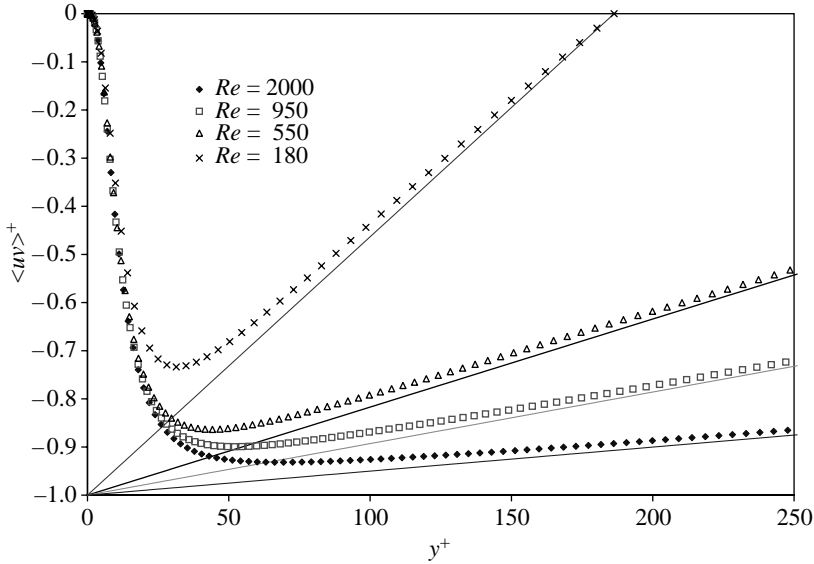


Figure 1. Plots of $\langle uv \rangle^+$ and $y^+/R^+ - 1$ for DNS of channel flow, $R^+ = 180, 550, 950$ and 2000 .

happen at ‘sufficiently’ high Reynolds number. But there is a problem which is seldom (if ever) addressed. It is obvious if equation (4.4) is integrated to obtain

$$u_*^2 = \left[-\langle uv \rangle + \nu \frac{\partial U}{\partial y} \right] - \frac{y}{\rho} \frac{dP_\infty}{dx}, \tag{4.5}$$

or in so-called inner variables, $y^+ = yu_*/\nu$ and $U^+ = U/u_*$

$$1 = -\langle uv \rangle^+ + \nu \frac{dU^+}{dy^+} + \lambda^+ y^+, \tag{4.6}$$

where λ^+ is a dimensionless pressure gradient defined as

$$\lambda^+ = \frac{\nu}{\rho u_*^3} \frac{dP_\infty}{dx}. \tag{4.7}$$

For a channel flow, the force exerted on the overall flow due to the streamwise pressure gradient is exactly balanced by the wall shear stress, so that

$$\frac{\tau_w}{\rho} = u_*^2 = -\frac{R}{\rho} \frac{dP}{dx}, \tag{4.8}$$

where R is the half-height of the channel. Therefore, for a channel, $\lambda^+ = 1/R^+$, where $R^+ = u_*R/\nu$. Thus, equation (4.6) becomes simply

$$1 = -\langle uv \rangle^+ + \nu \frac{dU^+}{dy^+} + \frac{y^+}{R^+}. \tag{4.9}$$

Since the viscous stress term is less than approximately 1% by $y^+ = 30$, this means that Reynolds shear stress drops linearly by 10% over the region of interest ($y/R \leq 0.1$), independent of the Reynolds number. Therefore, only in the innermost part of the constant stress layer can the pressure gradient be assumed negligible, and nowhere if the Reynolds number is not extremely high.

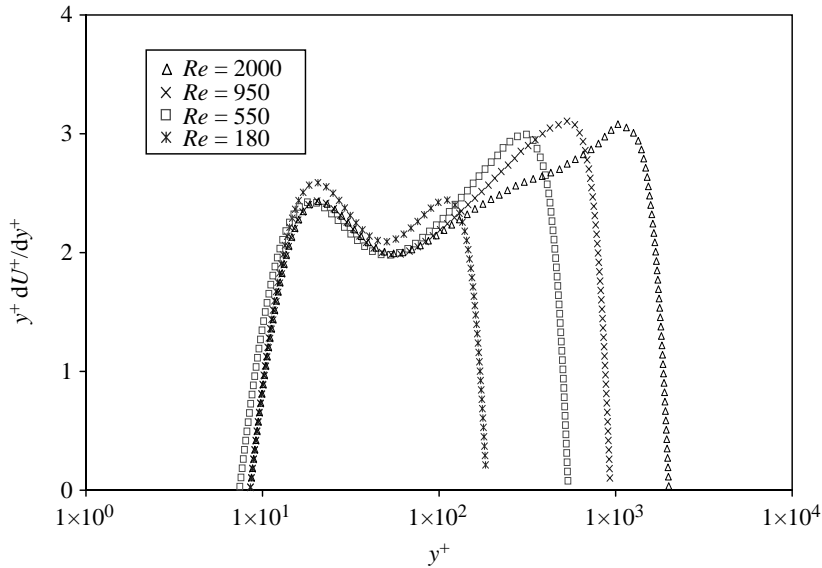


Figure 2. Plots of $y^+ dU^+/dy^+$ for DNS of channel flow, $R^+ = 180, 550, 950$ and 2000 .

The interplay of the Reynolds shear stress and the mean pressure gradient is shown in figure 1 using the recent DNS channel flow data from Jimenez and his co-workers (e.g. del Alamo *et al.* 2006). The values of R^+ are 180, 500, 950 and 2000. The linear drop of the total shear stress (viscous plus Reynolds) ensures that there is no constant stress region, since its contribution is more than 1% of the total beyond $y^+ = 2, 5, 10$ and 20 , respectively. And, contrary to popular belief, the situation will not improve with Reynolds number, since by $y^+ = 0.1R^+ - 0.2R^+$ (the approximate outer limit of the region of interest) the pressure gradient will always have reduced the Reynolds shear stress by 10%. Figure 2 shows that there is most certainly not a logarithmic region for these data. If the velocity profile were really logarithmic, the quantity plotted, $y^+ dU^+/dy^+$, would be constant over some region. Clearly it is not, at least for Reynolds number range of DNS data currently available.

By contrast, the boundary layer at zero pressure gradient does not have this pressure gradient problem, since the pressure gradient term is identically zero. If the Reynolds number is high enough for the convection (advection) and viscous terms to be negligible over some region (e.g. $\delta^+ \gg 300$), it truly does have a y -independent stress layer (even though it continues to vary slowly with x). Therefore, it *can* (at least in principle) behave like the assumed model equations (equation (4.1)), even if pipe or channel flow cannot. Boundary layers *with* pressure gradient, however, do not behave like either a zero-pressure-gradient boundary layer or a channel (or pipe) flow, since over the range logarithmic behaviour is expected, the role of the pressure gradient depends on the value of λ^+ . Therefore, the effect of the pressure gradient over the overlap region will in principle be different for each imposed λ^+ (with presumably different log parameters for each).

Thus, there is no reason *a priori* to believe boundary layers and pipes/channels to have identical inertial layers (or even mesolayers). The most that can be expected is that they might be identical only for the part of the flow which satisfies $y^+ \ll 0.1R^+$

or $y^+ \ll 0.1/\lambda^+$. And for most boundary-layer experiments, this is a very small region indeed. Therefore, while pipe/channel experiments (or DNS) may be of considerable interest in their own right, they cannot be a substitute for high Reynolds number boundary-layer experiments. This is especially true if the goal is to evaluate log constants or substantiate the log theory to within 10% (since their underlying equations differ over the overlap region by this amount).

5. Do pipe/channel flows have a logarithmic region anyway?

Now one might infer from the arguments above (and the DNS data) that pipe and channel flows should not have log layers (at least from the perspective of the eddy viscosity arguments), since there is really neither a constant Reynolds stress region nor any region where the pressure gradient can be ignored. We owe to Isakson (1937) and Millikan (1938) the original arguments that the contrary is true. The basic arguments have been presented in detail in many texts (e.g. Tennekes & Lumley 1972; Panton 1996; and in slightly more general form in Wosnik *et al.* 2000), and will only be summarized here. We will present them for pipe flow only, since experiments in pipes are much easier to realize at high Reynolds numbers than in channels, in part due to the difficulty of maintaining a two-dimensional mean flow in the latter. The superpipe data of Zagorala & Smits (1998), for example, go as high as $R^+ = 500\,000$. The basic theoretical arguments for channels, however, are the same.

The counterpart to equation (4.4) for fully developed flow in an axisymmetric pipe with smooth walls is

$$0 = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} r \left[-\langle uv \rangle + \nu \frac{\partial U}{\partial r} \right], \quad (5.1)$$

where r is measured from the pipe centreline and U , u and v are to be interpreted as corresponding to the streamwise and radial velocity components, respectively. Since P is nearly independent of r , we can multiply by r and integrate from the wall, R , to the running coordinate, r , to obtain the counterpart to equation (4.5) as

$$u_*^2 = \left(\frac{r}{R} \right) \left[-\langle uv \rangle + \nu \frac{\partial U}{\partial r} \right] - \frac{1}{2} \frac{(R^2 - r^2)}{R} \frac{1}{\rho} \frac{dP}{dx}. \quad (5.2)$$

Integration all the way to the centreline ($r=0$) yields the relation between the pipe radius, the wall shear stress and the imposed pressure gradient as

$$u_*^2 = -\frac{R}{2\rho} \frac{dP}{dx}. \quad (5.3)$$

Thus, of the four parameters in the equation, R , ν , $(1/\rho)dP/dx$ and u_*^2 , only three are independent. It follows immediately from dimensional analysis that the mean velocity for the entire flow can be written in either of the two ways

$$\frac{U}{u_*} = f_i(r^+, R^+), \quad (5.4)$$

and

$$\frac{U - U_c}{u_*} = f_o(\bar{r}, R^+), \quad (5.5)$$

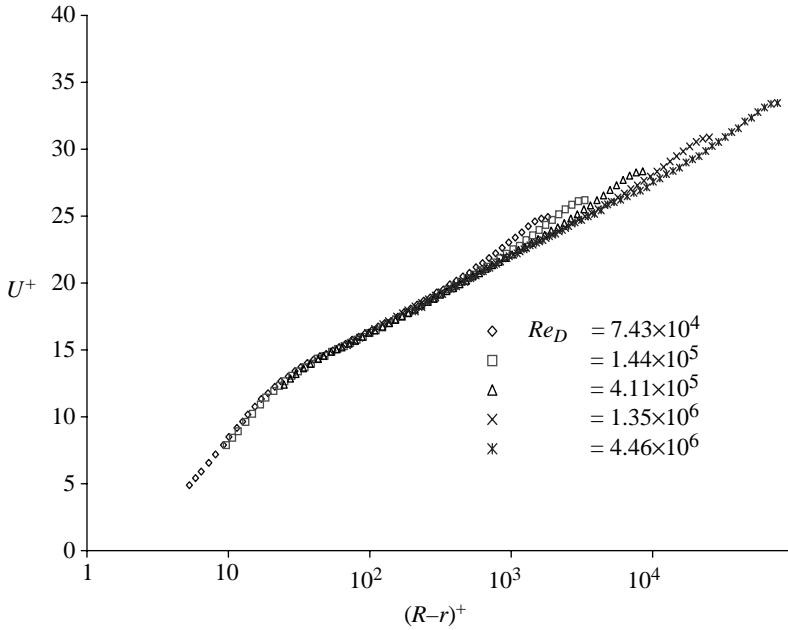


Figure 3. Five velocity profiles in inner variables. Superpipe data of McKeon *et al.* (2004a).

where U_c is the mean velocity at the centreline, $R^+ = u_* R/\nu$ is the ratio of outer to inner (or viscous) length-scales, $r^+ = ru_*/\nu$ is an inner normalization of r and $\bar{r} = r/R$ is an ‘outer’ normalization of r . Note that the velocity difference from the centreline, or ‘velocity deficit’, is used in the last expression to avoid having to take account of viscous effects as $R^+ \rightarrow \infty$.

Figures 3 and 4 show some of the recent superpipe data of McKeon *et al.* (2004b) plotted in inner and outer variables, respectively. (Note that the conventional labels inner and outer may appear opposite to the what they should be, since the outer is really the core region for the pipe and inner is a thin region closest to the pipe walls.) Clearly, the inner-scaled profiles appear to collapse near the wall, nearly collapse over a large intermediate range, and diverge when $(R-r)/R > 0.1$ or so. This means that the extent of the region of near-collapse in inner variables increases indefinitely as the Reynolds number, R^+ , increases. One might easily infer that in this region of near-collapse, the collapse will also improve to some asymptotic limiting profile (in inner variables). Similarly, the outer-scaled profiles appear to nearly collapse as long as $R^+ - r^+ > 300-500$ approximately, and again one could infer that the region of collapse might improve and continue all the way to the wall if the Reynolds number increased without bound.

We can define hypothetical inner and outer limiting profiles as $f_{i\infty}(r^+)$ and $f_{o\infty}(\bar{r})$, respectively, i.e.

$$\lim_{R^+ \rightarrow \infty} f_i(r^+, R^+) = f_{i\infty}(r^+), \tag{5.6}$$

$$\lim_{R^+ \rightarrow \infty} f_o(\bar{r}, R^+) = f_{o\infty}(\bar{r}). \tag{5.7}$$

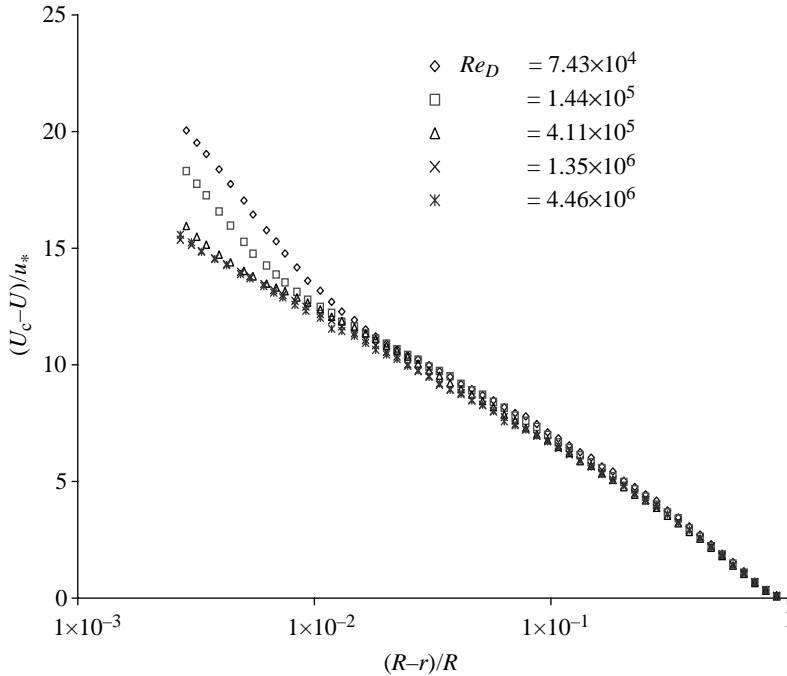


Figure 4. Five velocity profiles in outer variables. Superpipe data of McKeon *et al.* (2004a).

For finite values of R^+ , both equations (5.6) and (5.7) describe functionally the entire profile and R^+ acts as a parameter to distinguish the curves when they diverge. To see how they differ, consider the limit as $R^+ \rightarrow \infty$. Clearly, viscosity has disappeared entirely from $f_{o\infty}(\bar{r})$, so it can at most represent the mean velocity profile away from the near-wall region where viscous effects are not important. By contrast, $f_{i\infty}(r^+)$ can only describe the very near-wall region, since it has retained no information about R .

Now it is possible that the two limiting profiles, $f_{i\infty}$ and $f_{o\infty}$, do not link up, i.e. neither describes the flow far enough towards the pipe centre in the first case and towards the wall in the latter that they both describe a common region. But suppose they do (Millikan's great idea), so that both the inner and the outer scalings have a common (or overlap) region. Or thought of another way, can we 'stretch' the region over which the inner region collapses the data so that it overlaps a similar stretch in the other direction of the outer scaled version?

In fact, there are a variety of ways to show that the answer is yes. The traditional way is to set the inner limit of f_o equal to the outer limit of f_i and ask whether there can be a solution in the limit of infinite Reynolds number. Similar results can be obtained by matching derivatives in the limit (cf. Tennekes & Lumley 1972), or using matched asymptotic expansions (e.g. Panton 1996). Alternatively, Wosnik *et al.* (2000) used the methodology of near-asymptotics to seek *not an overlap region*, but instead a common region which survives at finite Reynolds number as the limits are approached. Regardless, *all* of the methodologies conclude that the mean velocity profile in the common

(or overlap) region should be logarithmic and given by the following equations:

$$\frac{U - U_c}{u_*} = \frac{1}{\kappa} \ln(1 - \bar{r} + \bar{a}) + B_o, \tag{5.8}$$

$$\frac{U}{u_*} = \frac{1}{\kappa} \ln(R^+ - r^+ + a^+) + B_i, \tag{5.9}$$

where $\bar{a} = a/R$, $a^+ = au_*/\nu$ and a is a spatial offset which is a necessary consequence of the need for invariance (cf. [Wosnik *et al.* 2000](#); [Oberlack 2001](#)). In addition, the friction and centreline velocities must be related by the following relationship (or friction law):

$$\frac{U_c}{u_*} = \frac{1}{\kappa} \ln R^+ + C, \tag{5.10}$$

where

$$C = B_i - B_o. \tag{5.11}$$

Thus, it is not enough to simply draw a logarithmic curve on a friction plot or an inner velocity plot and conclude anything more than that an empirical fit is possible. In fact, empirical log fits always seem to work, at least over some limited range, for just about any curve. Therefore, it is only when fits to all the three plots (friction, inner and outer mean velocity) can be linked together with common parameters using equations (5.8)–(5.11) that it can truly be concluded that pipe/channel flows are logarithmic and that theory and experiment agree.

The asymptotic theories conclude that κ , B_i and B_o must be constant, but only because the matching is done in the limit as $R^+ \rightarrow \infty$. Near-asymptotics, by contrast, tells how these limits are approached (inversely with powers of $\ln R^+$) and also how the different parameters are linked together, i.e. they must either be independent of R^+ or satisfy

$$\ln R^+ \frac{d}{d \ln R^+} (1/\kappa) = \frac{d}{d \ln R^+} (B_i - B_o). \tag{5.12}$$

But regardless of whether κ , B_i or B_o are constants (i.e. independent of Reynolds number) or only asymptotically constant, only two of them can be chosen independently.

So how well does this work? Quite well actually. [Figure 5](#) shows data for U_c/u_* from the superpipe experiments of [McKeon *et al.* \(2004a\)](#), along with several logarithmic fits to the data, both with average r.m.s. errors of about 0.2%. One curve uses constant values of κ and C , and another the variable Reynolds number version proposed by [Wosnik *et al.* \(2000\)](#). The values of κ and $C = B_i - B_o$ were 0.427 and 7.19 for the constant parameter (Reynolds number *independent*) analysis, while the limiting values for κ_∞ and $C_\infty = B_{i\infty} - B_{o\infty}$ for the Reynolds number *dependent* analysis were 0.429 and 7.96, respectively. The reason for the difference between the two values of C can be seen by examining the Reynolds number dependence in the [Wosnik *et al.*](#) theory for which

$$C_i = C_{i\infty} + \frac{(1 + \alpha)A}{(\ln R^+)^\alpha}, \tag{5.13}$$

$$\frac{1}{\kappa} = \frac{1}{\kappa_\infty} - \frac{\alpha A}{(\ln R^+)^{1+\alpha}}. \tag{5.14}$$

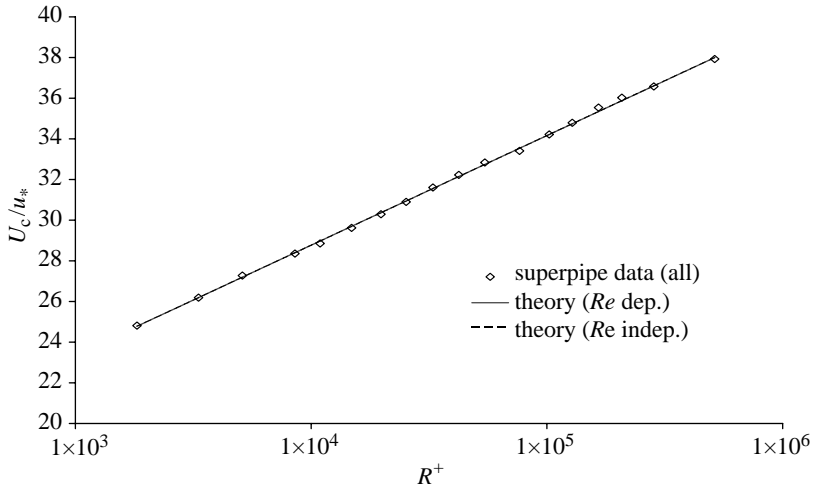


Figure 5. Plot of U_c/u_* versus R^+ using superpipe data of McKeon *et al.* (2004b).

Substituting these into equation (5.10) yields the refined friction law as

$$\frac{U_c}{u_*} = \frac{1}{\kappa_\infty} \ln R^+ + C_\infty + \frac{A}{(\ln R^+)^a}. \quad (5.15)$$

All of the extra Reynolds number dependence is in the last term of equation (5.15), and in fact it is this term which ‘adjusts’ the value from 7.96 to 7.19 over the range of the experiments. For this dataset, the optimal values of α and A were given by -0.932 and 0.145 , respectively, so the variation in κ over the entire range of the data was only from 0.426 to 0.427 . The corresponding variation of the last term in the friction law, however, was from -0.690 to -0.635 , enough to account for the slight lack of collapse of the mean velocity profiles in outer variables noted in figure 4.

Note that the constants determined by Wosnik *et al.* (2000) used an earlier (and ‘uncorrected’) form of the superpipe data, which showed a slightly different Reynolds number dependence. The differences are due to the static hole corrections in the new dataset. Unlike the conclusions from earlier versions of the superpipe data, there would appear to be little reason to consider the Reynolds number dependent version superior.

Table 1 summarizes the parameters from individual fits to five of the McKeon *et al.* (2004a) profiles selected to cover the entire range of the data. The value of κ determined from the friction data was taken as given, and the values of B_i and a were determined from a regression fit of each inner profile between $50 < (R-r)^+ < 0.1R^+$. The average r.m.s. errors are approximately 0.2% for all inner profiles. These same values, together with $C = B_i - B_o$, were then used to determine B_o , by optimizing the fit to the same profile in outer variables over the same range. The values of κ are remarkably constant, as are those of B_i . There might be a slight Reynolds number trend in the values of B_o . In view of the closest distance to the wall which can be measured (relative to a^+), the variation in a^+ is probably random positioning errors.

If the Reynolds number dependence is truly negligible, then the inner and outer mean velocity profiles should collapse when different Reynolds numbers are plotted together as in figures 3 and 4. The collapse of the profiles in inner

Table 1. Parameters for fits of log law to inner and outer profiles of [McKeon *et al.* \(2004a\)](#) using friction law values for Reynolds number dependent parameters.

| | | | | | |
|-----------------------|-------|-------|-------|------------------------|-------|
| $Re_D \times 10^{-5}$ | 0.743 | 1.44 | 4.11 | 13.5 | 44.6 |
| $R^+ \times 10^{-3}$ | 1.82 | 3.32 | 8.51 | 25.2 | 76.4 |
| κ | 0.426 | 0.426 | 0.427 | 0.427 | 0.427 |
| B_i | 5.62 | 5.50 | 5.64 | 5.87 | 5.85 |
| B_o | -1.65 | 1.77 | -1.65 | -1.43 | -1.46 |
| a^+ | -1.33 | -1.34 | -5.10 | -11.9 | -1.5 |
| % errin | 0.169 | 0.269 | 0.427 | 3.26×10^{-04} | 0.188 |
| % errout | 0.657 | 1.13 | 1.71 | 1.37 | 1.55 |

variables is excellent, consistent with the observations that κ is nearly constant, and B_i and a^+ are nearly so. The outer variable plot does not collapse quite so well, especially over the range for which the profiles are logarithmic. This lends support for the variable Reynolds number approach, which shows that the only significantly Reynolds number dependent parameter is B_o . This has implications for the asymptotic friction law, however, since the asymptotic values of $C = B_i - B_o$ are different, 7.19 versus 7.96.

So where does this leave us? These experiments (and most other pipe experiments as well) show an almost perfect agreement with the theoretical predictions, both the asymptotic and the near-asymptotic versions. Not only does there appear to be a region of logarithmic behaviour in the mean velocity profiles where we expected to find it ($30-50 < y^+ < 0.1-0.2R^+$), the parameters determined from fits to these and the logarithmic friction law satisfy the constraints among them. This is about the strongest experimental confirmation for a theory that can possibly be imagined.

So the analysis presented here (of part of the most recent version of the superpipe experiments) suggests strongly that the value of κ is approximately 0.43, a bit lower than the value of 0.44–0.45 suggested from the earlier uncorrected data and slightly higher than the estimate of [McKeon *et al.* \(2004a\)](#) of 0.42 using a larger set of the same data used herein. But all are higher than the earlier accepted value of 0.41, however, and most certainly not lower. The asymptotic value of the additive constant for the outer velocity profile (and friction law) can still be debated, but this debate in no way detracts from the overall conclusion. In spite of the absence of a constant total stress region (and hence the lack of validity of the early arguments for it), the logarithmic theory for a pipe flow can be taken as fact. One can infer this is probably also true for the channel, once data at sufficiently high Reynolds number becomes available to test it.

6. Boundary layers

As noted in the introduction, the differences in skin-friction drag on a modern plane resulting from $\kappa = 0.38$ and 0.43 are economically quite important. Since the assumption has been challenged that the near-wall regions of boundary layers and pipes were equivalent (as least as far as the logarithmic drag and velocity profiles were concerned), it is quite important to establish exactly how boundary layers do behave. Boundary-layer developments over the past decade have been

discussed in detail in a recent paper (George 2006) from the perspective of the ideas presented here, so the very brief summary below will not suffice for a careful reading of that paper. The primary concerns are twofold: first, the apparent lack of a consistent theory for the log law in boundary layers and second, the validity of the experiments. Both these are discussed below.

First, there is good reason to believe that the underlying log theory used above so convincingly for pipe/channel flow does not apply to boundary layers. The theory for pipe and channel flows depends crucially on the existence of the Reynolds number independent limits of the scaled profiles, equations (5.6) and (5.7). If the inner and outer profiles both do not scale with u_* , then there is no possibility of a logarithmic profile in the overlap region. Since the boundary layer is not homogeneous in the streamwise direction, there is no theoretical argument that can be used to justify an outer deficit law for boundary layers using u_* (as described in detail in George & Castillo 1997 and George 2006). In particular, substitution of equation (5.7) into the mean momentum equation (equation (4.3)) requires ignoring terms of the order u_*/U_∞ , while making inferences about skin friction which is of the order u_*^2/U_∞^2 . Thus, in the absence of supporting theory, any further inferences based on this deficit scaling are at most built on a foundation of empiricism, no matter how good the empirical collapse over some range of the data.

In spite of the theoretical objections, there is still evidence that a log friction law with $\kappa=0.38$ is an accurate description of at least the friction law for the boundary layer (e.g. Österlund 2000; Nagib *et al.* 2004), and perhaps even the velocity profiles. In an effort to resolve the apparent paradox, George (2006) suggested these might represent the leading terms in a logarithmic expansion of the power law solutions, and showed that a value near 0.38 was consistent with the power law coefficients. In fact, the dependence of the skin friction (or equivalently, u_*/U_∞) on Reynolds number (or δ^+) that results from a power law theory (which is theoretically defensible from first principles, George & Castillo 1997) is virtually indistinguishable from the log law fits to the same power law theoretical curves.

Whatever the reason for the apparent success of the log law in zero-pressure-gradient boundary layers, in the absence of a consistent log theory for the boundary layer (or any developing wall-bounded flow), there is no reason to believe that logarithmic friction and velocity laws for boundary layers should be linked to those for pipes and channels, no matter how good the empirical curve fits. The consequences of this are quite important, since it means that boundary layers could be quite well described by logarithms with $\kappa=0.38$, independent of all other considerations.

Second, there are reasons to believe that there are significant problems with at least some of the boundary-layer mean velocity measurements that have been used to argue for the log law with $\kappa=0.38$. George (2006) pointed out that the recent results from Nagib *et al.* (2004) are not consistent with the momentum integral, differing by as much as 30–40%. Thus, either the flow is not a two-dimensional incompressible smooth wall turbulent boundary layer, or the skin-friction and/or velocity measurements are in error.

George (2006) also considered in detail the other extensive and relatively recent set of mean velocity measurements by Österlund (2000). Contrary to the claims made by Österlund *et al.* (2000), these data were shown to be equally

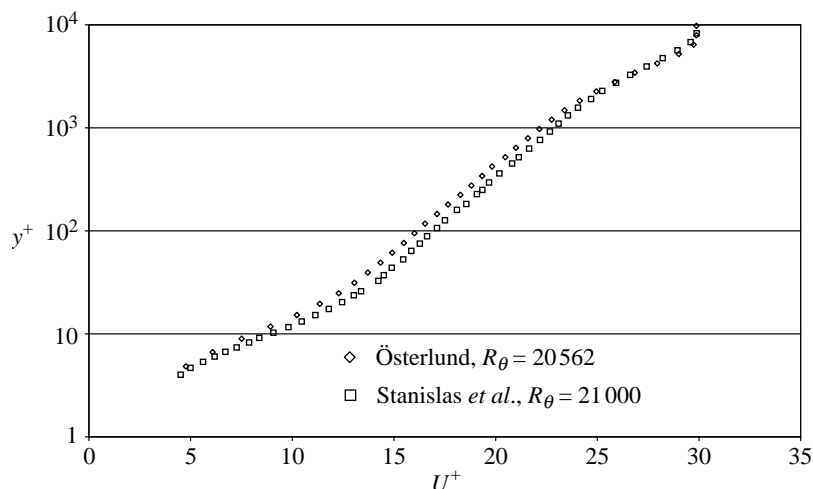


Figure 6. Comparison of mean velocity profiles from Österlund (2000) at $R_\theta=20\,562$ and M. Stanislas *et al.* (2006, personal communication) at $R_\theta=21\,000$.

consistent with either log or power law curve fits, and in fact the curve fits were indistinguishable. It was also pointed out that in the absence of Reynolds stress measurements, there was no way to confirm that any of the measured profiles were consistent with the mean equations of motion. This was of considerable concern, since unlike earlier boundary data (e.g. Smith & Walker 1959), the Österlund data showed virtually no Reynolds number dependence in the overlap region, but did in the outer region of the flow (where one would least expect to find it).

The aforementioned concern about the Österlund experiment was considerably heightened by recent results from ongoing experiments at the Lille boundary-layer facility (mentioned above) by M. Stanislas *et al.* (2006, personal communication; see Carlier & Stanislas 2004), which became available to me in the course of preparing this paper. Their mean velocity profile obtained at $R_\theta=21\,000$ is plotted in figure 6 along with the corresponding profile from Österlund (2000). The friction coefficients for the two experiments were almost exactly the same (meaning the normalized shear stresses were in agreement), as is evident from the near overlay of the curves at the largest distances from the wall (effectively U_∞/u_*). This is encouraging, since the Österlund shear stress was estimated using an oil film method and the Lille result obtained by micro-PIV. Incidentally, the latter value differs by approximately 3.5% from the shear stress estimated using the Clauser chart on the Lille data, a substantial difference in view of the questions being asked.

By contrast to the data at large distances from the wall, the mean velocity profiles near the wall (inside $0.15\delta_{99}$ or $y^+ < 1200$) differ substantially until they come together again inside $y^+=10$. Figure 7 shows a linear-linear plot of both sets of data showing only the region from $100 < y^+ < 1100$. The two profiles appear virtually identical over this range, but shifted in both velocity and position. This is difficult to understand. Both sets of measurements were obtained using hot-wire anemometry, so there is no obvious reason for the

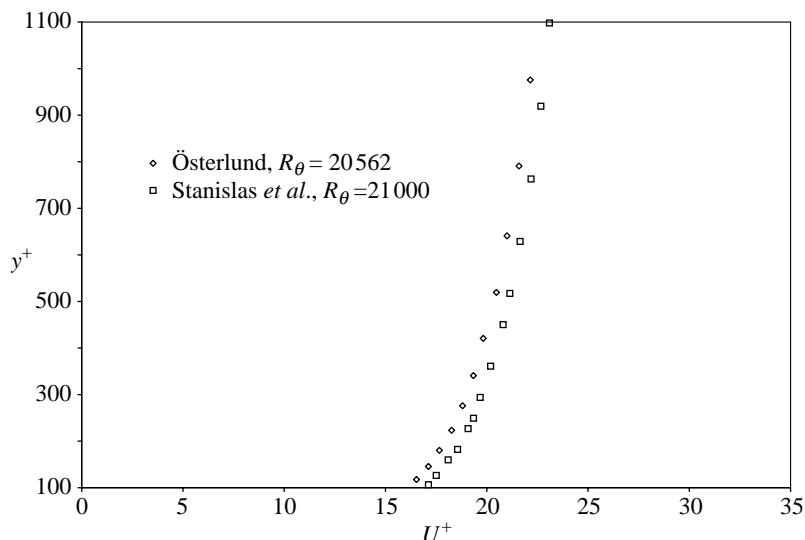


Figure 7. Expanded linear-linear plot of data in figure 6 from $100 < y^+ < 1100$.

difference. The Lille profiles, however, were confirmed by more sparsely spaced PIV measurements. Moreover, the Lille profiles are consistent with the measured Reynolds stress profiles (from PIV) and the differential equations of motion.

Both the log profile and the power law profile, $U^+ = C_i(y^+ + a^+)^{\gamma}$, can be optimized to fit the Lille velocity profiles to within 0.2% for $50 < y^+ < 0.1\delta^+$. This was no surprise, since, as pointed out by George (2006), the functional forms are indistinguishable, at least over the range of data available. The results for γ , C_o , C_i and a^+ were 0.119, 0.97, 9.87 and -8.3 , respectively, different from the earlier estimates of George & Castillo (1997) as expected since the data were different from those previously considered. The values for the logarithmic fit, on the other hand, were quite surprising given the difference in the measured profiles, since the optimal values were 0.384, 4.86, -2.03 and 4 for κ , B_i , B_o and a^+ , respectively. By comparison, the log values for the Österlund profile were $\kappa=0.384$, $B_i=4.16$ and $a^+=0$.

In other words, the values of κ from the log curve fits to the Stanislas *et al.* and Österlund experiments were identical, even though the profiles and other constants differed substantially. Thus, in spite of the differences and shortcomings of the various experiments (which remain to be explained and reconciled), there would appear to be increasing evidence for $\kappa=0.38$ (or even 0.384) for boundary layers.

7. Summary and conclusions

Therefore, in summary, there is no justification, theoretical or experimental, for a *universal* log law for all wall-bounded flows, no matter how aesthetically appealing or potentially useful an idea. At very least, boundary layers and pipe/channel flows are fundamentally different. Or viewed another way, the log law represents the inertial region of pipes, channels and boundary layers to about

the same degree that their underlying equations have the same terms, which is to within approximately 10%. Thus, the historical value of $\kappa=0.41$ is probably best seen as a compromise for different flows, accurate to within these limits.

The log theory does apply quite rigorously to pipe flows with $\kappa=0.43$ and perhaps to other wall-bounded flows homogeneous in horizontal planes (e.g. channels, Couette flow, the neutral planetary boundary layer, etc.). But it is a power law theory for the boundary layer that can be derived from first principles using equilibrium similarity analysis and near-asymptotics. This theory predicts (without additional assumptions) a number of things that have also been observed, but which require additional hypotheses with a log theory. Among them are differing outer scales for the normal and shear stress components (U_o^2 and u_*^2 , respectively), the consequent dependence of the turbulence properties of boundary layers in the overlap region on mixed scales and the dependence of pressure fluctuations on the ratio U_o/u_* . Moreover, the same principles can be used to predict different results for different flows (like wall jets and boundary layers with pressure gradient), again as observed.

Nonetheless, theoretical arguments notwithstanding, the log ‘law’ also appears to apply to developing boundary layers. If not the leading term in a logarithmic expansion of the power law solution, it is at least a local and empirical description. And to the degree that developing boundary layers can be described this way, the value of κ for them is approximately 0.38.

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