Similarity Theory for Forced Convection Thermal Boundary Layers

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ABSTRACT

A new theory for the forced convection thermal boundary layer is outlined. The inner and outer scaling parameters are derived by applying the Asymptotic Invariance Principle to the thermal boundary layer equations. The temperature profiles for the overlap region and the heat transfer law are derived using Near-Asymptotics, and are shown to be of power law form. A possible form for the dependence of the parameters on the local Reynolds number is suggested by analogy with the momentum boundary layer.

INTRODUCTION

Recently, George and Castillo 1997 have presented new similarity analyses for isothermal boundary layers (see also George et al. 1996, Castillo 1997). The purpose of this work is to extend this new understanding to the thermal boundary layer of forced convection.

The essential difference between the classical analysis of turbulent momentum boundary layers and the new theory stems from the form of the velocity deficit law for the outer flow. In the old theory, the velocity deficit, $U - U_\infty$ is scaled with the friction velocity, u_*, on entirely empirical grounds. The new theory begins by applying an Asymptotic Invariance Principle which demands that any proper scaling reduce in the limit of infinite Reynolds number to a similarity solution of the outer (or inner) boundary layer equations. Only by scaling the velocity deficit with U_∞ can a properly invariant solution be obtained. There were several consequences of this: First, for finite Reynolds number, no scaling can collapse the data perfectly, but only produce a trend toward an asymptotic state. Second, the velocity profiles in the overlap region were power laws and not logarithms, as previously believed.

Castillo 1997 extended the theory to include boundary layers with imposed pressure gradient. These boundary layers were shown to be uniquely described by a single equilibrium parameter, $\Lambda = [\delta/\rho U_\infty^2 d\delta/dx] dP_\infty/dx$ which reduces to the Clauser parameter in the limit of infinite Reynolds number. Surprisingly (and contrary to earlier attempts to analyze such boundary layers), nearly all the measurements either conform to the new equilibrium definition, or show evidence or moving from one equilibrium state to another. Thus similarity techniques (at least of the type employed here) seem to describe a large portion of the available measurements, contrary to the conventional wisdom.

In the following sections, the results for the momentum boundary layer will be briefly reviewed, and a new theory for the thermal boundary layer will be outlined.

REVIEW OF THE MOMENTUM BOUNDARY LAYER RESULTS

George and Castillo 1997 show that the *Asymptotic Invariance Principle (AIP)* requires that properly scaled profiles reduce to similarity solutions of the inner and outer equations separately in the limit of infinite Reynolds number. The inner profiles scale with ν and u_* as in the classical analysis; i.e.,

$$\frac{U}{u_*} = f_i\left(\frac{yu_*}{\nu}, \delta^+\right) \quad (1)$$

The presence of the local Reynolds number, $\delta^+ =$

$u_*\delta/\nu$, means that the inner scaling will reduce to the Law of the Wall only in the limit as $\delta^+ \to \infty$, and there will be a Reynolds number dependence for all finite values outside the linear sublayer. Moreover, the inner scaled profile describes the entire boundary layer, except in the limit where it loses the ability to describe the outer flow.

The outer flow requires two velocity scales, u_* and U_∞, in addition to δ, the boundary layer thickness. The velocity deficit law, in particular, scales with U_∞ instead of u_* as in the classical analysis; i.e.,

$$\frac{U - U_\infty}{U_\infty} = f_o\left(\frac{y}{\delta}, \delta^+\right) \quad (2)$$

while the Reynolds shear stress scales with u_*^2. Again, the presence of δ^+ means that the outer scaling collapses the data only in the limit as $\delta^+ \to \infty$. Also, the outer scaled profile describes the entire boundary layer for finite δ^+, and loses the ability to describe the inner layer only in the limit.

The fact that both inner and outer scaled profiles describe the entire boundary layer for finite values of δ^+, but reduce to inner and outer profiles in the limit, is used to determine their functional forms in the "overlap" region which both retain. Since the ratio of scales, u_*/U_∞, is Reynolds number dependent, so must be the "overlap" region. By using Near-Asymptotics it is possible to show that to leading order the profiles in this overlap region are power laws in $y + a$ where a represents an origin shift; i.e.,

$$\frac{U - U_\infty}{U_\infty} = C_o\left(\frac{y+a}{\delta}\right)^\gamma \quad (3)$$

$$\frac{U}{u_*} = C_i\left(\frac{y+a}{\eta}\right)^\gamma \quad (4)$$

The friction law is also a power law in δ^+ and is given by

$$\sqrt{\frac{c_f}{2}} = \frac{u_*}{U_\infty} = \frac{C_o(\delta^+)}{C_i(\delta^+)}\delta^{+^{-\gamma(\delta^+)}} \quad (5)$$

The three parameters in these power laws depend on δ^+ (or R_θ), but are asymptotically constant and are linked by a constraint equation.

Application of the AIP to the Reynolds stress equations is used to determine the proper scaling laws for many of the turbulence quantities in the inner and outer layers. A surprising result is that in the outer layer (and overlap region) some of them scale with both u_* and U_∞, hence they are never Reynolds number independent, except in the limit as $\delta^+ \to \infty$. Other consequences of the theory are that the asymptotic values of δ_*/δ and θ/δ are non-zero constants. By considering when the Reynolds stress and energy dissipation become Reynolds number independent, it is possible to estimate that the asymptotes for c_f, $d\theta/dx$, and γ should not be reached until $R_\theta > 10^5$, which is beyond the range of existing experiments.

It is argued that there must exist a *mesolayer* in the region approximately defined by $30 < y^+ < 300$ in which the energy dissipation evolves from $\nu q^2/y^2$ to q^3/y. In this mesolayer, the Reynolds stress and mean flow equations retain a Reynolds number dependence, even though the terms explicitly containing the viscosity are negligible. The parameter a which appears in the overlap solutions is shown to account for this effect. Because of the mesolayer (and a^+), the velocity profile is a simple power law in y only beyond $y^+ > 300$, well outside where it has commonly been sought. The region between $300 < y^+ < 0.1\delta^+$ (approximately) is identified as the *inertial sublayer*.

The abundant experimental data were shown to be in reasonable agreement with the new theory, as long as the experimental values of the shear stress were NOT inferred using an assumed log profile. Further it was shown that all of the data can be described with just three constants and a semi-empirical equation for the variation of γ with δ^+ given by

$$\gamma - \gamma_\infty = \frac{\alpha A}{(\ln \delta^+)^{1+\alpha}} \quad (6)$$

It followed from a constraint equation that

$$\frac{C_o}{C_i} = \frac{C_{o\infty}}{C_{i\infty}} \exp[(1+\alpha)A/(\ln \delta^+)^\alpha] \quad (7)$$

and

$$\frac{u_*}{U_\infty} = \frac{C_{o\infty}}{C_{i\infty}}\delta^{+^{-\gamma_\infty}} \exp[A/(\ln \delta^+)^\alpha] \quad (8)$$

The optimal values obtained for the constants were $C_{o\infty} = 0.897$, $C_{i\infty} = 55$, $\gamma_\infty = 0.0362$, $A = 2.90$, and $\alpha = 0.46$. The data for C_o were well described over the entire range from $465 < R_\theta < 48,292$ by the expression,

$$\frac{C_o}{C_{o\infty}} = 1 + 0.283 \exp(-0.00598\delta^+) \quad (9)$$

Finally, the same approach was applied to flows which are homogeneous in the streamwise direction, like fully-developed channels and pipes. The result was overlap solutions which are *not* power laws, but instead logarithmic functions of the variable $y + a$. Like the boundary layer, however, the parameters are Reynolds number dependent and only asymptotically constant. The theory is in excellent agreement with recent pipe flow experiments at very high Reynolds numbers. The experiments, together with the theory, suggest $\kappa \to 0.447$.

THE THERMAL BOUNDARY LAYER EQUATIONS

Consider the thermal boundary layer formed by flow over a heated surface. The effects of buoyancy will be assumed negligible, as will those due to the variation of thermal properties with temperature. The temperature effectively acts as a passive scalar entirely driven by conduction and the velocity field.

It is well-known that at high Reynolds and Peclet numbers, the conduction term is negligible over most of the boundary layer, so the outer thermal equation reduces to

$$U\frac{\partial(T-T_\infty)}{d\partial x}+V\frac{\partial(T-T_\infty)}{\partial y}=-\frac{\partial < vt >}{\partial y} \tag{10}$$

where the ambient field is assumed to be at uniform temperature and can thus be subtracted.

Near the wall the conduction term must remain so that the wall boundary condition can be satisfied. A consequence of the re-scaling necessary to keep it is that the convection terms can be shown to be negligible near the near wall, so the thermal equation there reduces to

$$0=\frac{\partial}{\partial y}\Big[-< vt > +\alpha\frac{\partial(T-T_\infty)}{\partial y}\Big] \tag{11}$$

The inner and outer equations, equations 10 and 11, are exact in the limit as the Reynolds and Peclet numbers go to infinity.

Equation 11 can be integrated from the wall to yield

$$F_w=-< vt > +\alpha\frac{\partial(T-T_\infty)}{\partial y} \tag{12}$$

where F_w is the flux parameter defined by

$$F_w=-\frac{q_w}{\rho C_p} \tag{13}$$

and q_w represents the wall heat flux. In general, q_w and T_w, the wall temperature are x-dependent.

SIMILARITY ANALYSIS OF THE NEAR WALL REGION

The Asymptotic Invariance Principle (A.I.P.) requires that properly scaled inner and outer profiles reduce to similarity solutions of the inner and outer equations in the limit as Reynolds and Peclet numbers go to infinity (i.e., the limit in which the equations themselves are valid). Thus for the inner equations solutions are sought of the form

$$T-T_\infty=T_{si}g_i({y^+}_T,\delta_T^+,Pr) \tag{14}$$

$$-< vt >=F_{si}h_i({y^+}_T,\delta_T^+,Pr) \tag{15}$$

where

$${y^+}_T\equiv\frac{y}{\eta_T} \tag{16}$$

and η_T is a length scale which remains to be determined. The parameter δ_T^+ is the ratio of outer to inner length scales and remains to be defined. It can be shown *a posteriori* that $\delta_T^+\to\infty$ as $R_x\to\infty$. The non-zero boundary conditions are given by

$$T-T_\infty=T_w-T_\infty\equiv\Delta T_w \tag{17}$$

at $y=0$, and

$$-\overline{vt}\to F_w \tag{18}$$

as $y\to\infty$. The last condition is only true in the limit as $\delta_T^+\to\infty$, which insures that the outer flow is never reached so the inner equation is never invalid (at least in inner variables).

Substitution into equation 12 and dividing by F_w shows that similarity solutions are possible only if

$$F_{si}\sim F_w, \tag{19}$$

$$T_{si}\sim\Delta T_w \tag{20}$$

and

$$\eta_T\sim\frac{\alpha\Delta T_w}{F_w} \tag{21}$$

Note that the symbol '$\sim$' does not mean order of magnitude, but rather 'has the same x-dependence as'. These can be taken as equalities without loss of generality, and thus the inner scales are defined.

These results are quite different from the classical law of the wall for thermal boundary layers; here it is deduced and does not depend simply on dimensional analysis or empirical scaling. Note that as long as the argument δ_T is finite, the profile of equation 14 is simply a scaled temperature profile for the entire boundary layer. Only in the limit of infinite δ_T^+ does it lose the ability to describe the outer flow and become an inner solution.

Figure 1 shows six of the temperature profiles measured by Reynolds et al. 1958 normalized in the inner variables derived here. The data were taken at two different values of the free stream velocity for three positions along the surface. Note the excellent collapse near the wall for all the data, the slight dependence on Reynolds number in what will be identified later as the overlap region, and the departure from similarity in the outermost region. Particularly noteworthy is the dramatic difference between the two sets of measurements outside the near wall region, clearly suggesting a dependence on initial conditions.

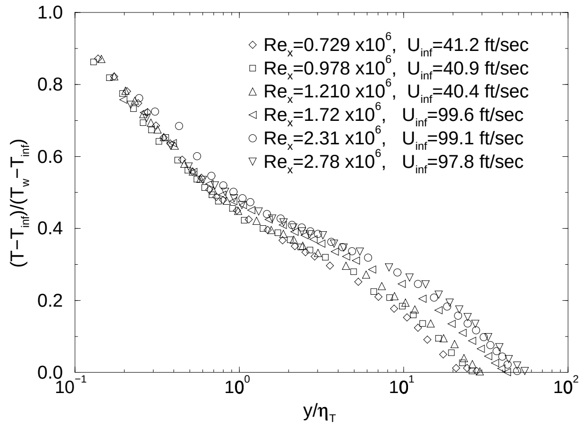

Figure 1: Mean temperature data in inner variables. Data of Reynolds *et al.* 1958

SIMILARITY ANALYSIS OF THE OUTER FLOW

Application of the A.I.P. to the outer flow is carried out in the same manner. Solutions are sought of the form

$$T - T_\infty = T_{so} g_o(\overline{y}_T, \delta_T^+, Pr) \quad (22)$$

$$- < vt > = F_{so} g_i(\overline{y}_T, \delta_T^+, Pr) \quad (23)$$

where

$$\overline{y}_T = y/\delta_T \quad (24)$$

and δ_T remains to be determined. Note that the dependence of the outer solutions on δ_T^+ might be expected to vanish in the limit as $\delta_T^+ \rightarrow \infty$, and that of the Prandtl number as well. The result for the outer velocity from George and Castillo 1997 must also be utilized; it is

$$U = U_\infty[1 + f_o(\overline{y}, \delta^+)] \quad (25)$$

where

$$\overline{y} = \frac{y}{\delta} \quad (26)$$

Substition into equation 10 and dividing by $U_\infty T_{so}/\delta_T$ shows that the necessary conditions for similarity are:

$$\frac{d\delta}{dx} \sim \frac{d\delta_T}{dx} \sim \frac{F_{so}}{U_\infty T_{so}} \sim \frac{\delta_T}{T_{so}} \frac{dT_{so}}{dx} \sim \frac{\delta}{U_\infty} \frac{dU_\infty}{dx} \quad (27)$$

The last condition involving U_∞ applies only if the free stream speed is varying (i.e., $dP_\infty/dx \neq 0$).

Thus it is clear that both the thermal and momentum boundary layer thicknesses have the same x-dependence, and can at most differ by a constant (or alternatively, have different virtual origins). Note that there is no suggestion in either the outer thermal or the momentum boundary governing equations that the outer flow might be independent of initial (or upstream) conditions.

Also it is immediately obvious that

$$F_{so} \sim U_\infty T_{so} \frac{d\delta}{dx} \sim U_\infty T_{so} \frac{d\delta_T}{dx} \quad (28)$$

This condition on T_{so} will be examined later and shown to yield the outer temperature scale.

THE OUTER TEMPERATURE SCALE

Equation 28 is the thermal counterpart to the similarity condition on the Reynolds stress in the outer layer of the momentum boundary layer; namely, $R_{so} \sim U_\infty^2 d\delta/dx$. Since the heat flux across the inner layer is constant and equal to F_w (but only in the limit of infinite Peclet number), it provides an inner boundary condition on the heat flux, so

$$F_{so} \sim F_w \quad (29)$$

The coefficient in equation 29 is dependent on Peclet number, but asymptotically constant. A similar argument for the Reynolds stress in the outer layer yielded

$$R_{so} \sim U_\infty^2 \frac{d\delta}{dx} \sim u_*^2, \quad (30)$$

again with a Reynolds number dependent coefficient which was constant in the limit.

It follows immediately from equations 27, 29 and 30 that the outer temperature scale which reduces to the proper limit (in the limit of infinite Peclet number) is given by

$$T_{so} = T_{si} \frac{U_\infty}{u_*} = \frac{F_w}{u_*} (\frac{U_\infty}{u_*}) \quad (31)$$

Thus

$$\frac{T_{so}}{T_{si}} = \frac{F_w}{u_* \Delta T_w} \frac{U_\infty}{u_*} \equiv H \quad (32)$$

The parameter H can easily be shown to be related to the Stanton number star, St_*, and the Stanton number, St, by

$$H = St_* \sqrt{\frac{2}{c_f}} = \frac{2St}{c_f} \quad (33)$$

where c_f is the friction coefficient.

Figure 2 shows the same data above normalized in outer variables. The value of δ_T was taken as the y-location where the temperature had dropped to $0.10(T_w - T_\infty)$. The friction velocity was computed from the momentum integral equation, which included a correction for the slight adverse pressure gradient in the tunnel. The data clearly collapse well over the outer part of the flow as long as the upstream conditions are fixed. They show the same differences in the overlap region for the two different sets, however, just as for the inner scaled profiles above. Whether this is a consequence of the Reynolds number dependence of the overlap region, or the effect of upstream conditions must be a matter for future investigations to determine.

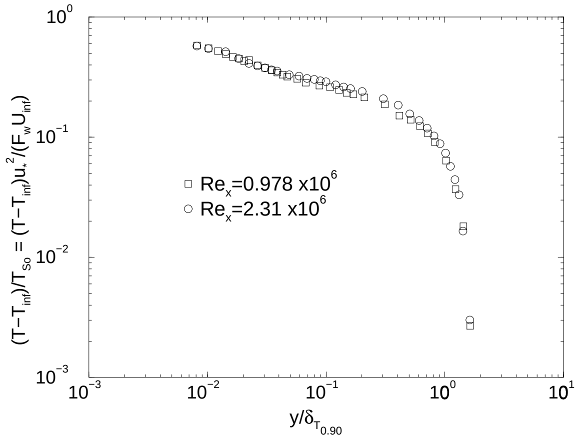

Figure 2: Mean temperature data in outer variables. Data of Reynolds *et al.* 1958

THE OVERLAP REGION OF THE TEMPERATURE PROFILES

In the limit as the Peclet number becomes infinite, the ratio of outer to inner length scale also becomes infinite. Thus, although at finite Peclet numbers, both of the scaled profiles of equation 14 and 22 can describe the temperature profile everywhere (because of their dependence on δ_T^+), in the limit the inner cannot describe the outer region, nor the outer the region closest to the wall. There might, however, exist an overlap region in which both inner and outer scalings work in the limit. This can indeed be shown to be the case by the methodology of George and Castillo 1997 which makes clear the Reynolds and Prandtl number dependence of the parameters.

It follows after some manipulation of the equations that the inner temperature profile is given by

$$g_i(y^+{}_T, \delta_T^+, Pr) = A_i(y^+{}_T + a^+)^{\gamma_T} \tag{34}$$

where $A_i = A_i(\delta^+, Pr)$ and $\gamma_T(\delta_T^+)$ is defined as for the velocity profile by

$$\gamma_T \equiv \frac{\delta_T^+}{H}\frac{dH}{d\delta_T^+} \tag{35}$$

The parameter a arises from the fact that the results must be independent of the origin in y. The exact behavior of a must be established from experiment (at least at this point), but it is certainly related to the mesolayer and the effects of the low turbulence Reynolds number on the thermal dissipation. For the momentum boundary layer, $a^+ \approx -16$, so that it might be constant for the thermal layer as well.

The outer temperature profile can similarly be shown to be given by

$$g_o(\overline{y}_T, \delta_T^+) = A_o(\overline{y}_T + \overline{a})^{\gamma_T} \tag{36}$$

where $A_o = A_o(\delta_T^+, Pr)$.

Thus the temperature profile in the overlap region is described by a power law, but not a simple one because of the dependence of the coefficients on δ_T^+. In the limit as $\delta_T^+ \to \infty$, $A_i \to A_{i\infty}$ and $A_o \to A_{o\infty}$. The AIP requires that both $A_{i\infty}$ and $A_{o\infty}$ be asymptotically finite and non-zero, just as for the velocity profile. Moreover, the limiting value of γ_T must be zero to insure a finite local dissipation of temperature fluctuations.

The parameters A_i, A_o, and γ_T must also satisfy a constraint equation given by

$$\ln \delta_T^+ \frac{d\gamma_T}{d\ln\delta_T^+} = \frac{d\ln A_o/A_i}{d\ln\delta_T^+} \tag{37}$$

which is similar to the analogous requirement for the momentum boundary layer parameters.

The log-log plot of Figure 2 gives a clear clue of the presence of power law behavior of the type suggested here, including the Reynolds number dependence. As noted above, because of the mesolayer and the possible origin offset, determination of power law behavior is more complicated than simply drawing a straight line on a log-log plot (cf. George and Castillo 1997). The exact nature of this region will be explored in detail in future publications.

THE HEAT TRANSFER LAW

The heat transfer law governing the forced boundary layer can be derived readily by substituting the overlap solutions for g_i and g_o into equations 14 and 22, then equating the temperatures. The result is:

$$H = \frac{T_{so}}{T_{si}} = \frac{A_i}{A_o}\delta_T^{+\gamma_T} \tag{38}$$

or

$$St_* \frac{U_\infty}{u_*} = \frac{A_i}{A_o}\delta_T^{+\gamma_T} \tag{39}$$

or using equation 5

$$St_* = \frac{A_i}{A_o}\frac{C_o}{C_i}\delta_T^{+\gamma_T}\delta^{+-\gamma} \quad (40)$$

It is clear from equation 40 that the exact behavior of the Stanton number based on the friction velocity is an interesting interaction between the momentum and thermal boundary layer parameters. If $\gamma_T > \gamma$, $St_*\sqrt{2/c_f}$ increases as the boundary layer grows, whereas $St_*\sqrt{2/c_f}$ decreases if $\gamma_T < \gamma$. On the other hand, if $\gamma_T = \gamma$, then $St_*\sqrt{2/c_f}$ does not change downstream. There is reason to suspect that this may indeed be the case, at least when both boundary layers have the same effective origin. If so, then the Stanton number is proportional to the friction coefficient, at least asymptotically, just as in the classic Reynolds analogy. These possibilities will be explored in subsequent papers.

THE VARIATION OF THE PARAMETERS WITH δ_T^+

A solution to equation 37 can be written as

$$\frac{A_o}{A_i} = \exp[(\gamma_T - \gamma_{T\infty})\ln\delta_T^+ + h_T] \quad (41)$$

where the single unknown new function h_T is at most a function of δ_T^+ and the Prandtl number. It follows immediately from equations 35 and 38 that

$$\gamma_T - \gamma_{T\infty} = -\frac{dh_T}{d\ln\delta_T^+} \quad (42)$$

and

$$H = \delta_T^{+\gamma_{T\infty}} e^{-h_T} \quad (43)$$

Thus the entire problem is reduced to finding either theoretically or empirically the function h_T.

It may be possible to proceed by analogy with the momentum boundary layer and take

$$h_T - h_{T\infty} = \frac{A_T}{(\ln\delta_T^+)^{\alpha_T}} \quad (44)$$

where $h_{T\infty} = \ln A_{o\infty}/A_{i\infty}$. If so, then

$$\gamma_T - \gamma_{T\infty} = \frac{\alpha_T A_T}{(\ln\delta_T^+)^{1+\alpha_T}} \quad (45)$$

$$\frac{A_o}{A_i} = \frac{A_{o\infty}}{A_{i\infty}}\exp[(1+\alpha_T)A_T/(\ln\delta_T^+)^{\alpha_T}] \quad (46)$$

and

$$H = \frac{A_{i\infty}}{A_{o\infty}}\delta_T^{+\gamma_{T\infty}}\exp[-A_T/(\ln\delta_T^+)^{\alpha_T}] \quad (47)$$

The heat transfer law and overlap profiles are then entirely specified by $\gamma_{T\infty}$, $A_{i\infty}$, $A_{o\infty}$ and the constants A_T and α_T. It will not be too surprising if some of these are the same as for their counterparts in the momentum boundary layer.

Future work will attempt to establish the exact form of h_T and determine the necessary constants. In addition, the implications for the scaling of the fluctuating quantities will be explored.

ACKNOWLEDGEMENTS

The authors are grateful to M.M. Gibson and J. Knoell for their helpful comments.

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