LOCALLY AXISYMMETRIC TURBULENCE

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Abstract

The failure of local isotropy to describe the experimentally obtained derivative moments in turbulent shear flows has previously been well-documented, but is briefly reviewed. The same data is then used to evaluate the hypothesis that the turbulence is *locally axisymmetric*. *Locally axisymmetric* turbulence is defined herein as turbulence which is locally invariant to rotations about a preferred axis.

The derivative moment relations are derived from the general form of the two-point velocity correlation tensor near the origin for axisymmetric turbulence. These are used to derive relations for the rate of dissipation of kinetic energy, the mean square vorticity, and the components of each. Almost all of the experimental derivative moment data are shown to be consistent with these equations, and thus with *local axisymmetry*.

1 Introduction

1.1 The Rate of Dissipation

The characterization and measurement of small scale motions of turbulent flow have represented two of the most challenging problems of turbulence research over the past 40 years. One of the primary experimental concerns has been the determination of the mean square gradients of the fluctuating velocity field which depend disproportionately on the small turbulence scales. These mean square gradients in turn determine the rate of dissipation of turbulence kinetic energy and the mean square vorticity.

For a Newtonian fluid with kinematic viscosity, ν, the rate of dissipation of turbulence kinetic energy per unit volume, ε, is defined by

$$\varepsilon = 2\nu \langle e_{ij} e_{ij} \rangle \tag{1}$$

where e_{ij} is the strain rate tensor. If u_i represents the component of velocity in the x_i direction, then e_{ij} is defined as

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2}$$

Using this, equation 1 can be expanded to yield

$$\varepsilon = \nu \left[\left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right\rangle + \left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right\rangle\right] \tag{3}$$

or

$$\begin{aligned} \varepsilon \;=\; & \nu \Big\{ 2\left[\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle + \left\langle\left(\frac{\partial u_2}{\partial x_2}\right)^2\right\rangle + \left\langle\left(\frac{\partial u_3}{\partial x_3}\right)^2\right\rangle\right] \\ & + \left[\left\langle\left(\frac{\partial u_1}{\partial x_2}\right)^2\right\rangle + \left\langle\left(\frac{\partial u_2}{\partial x_1}\right)^2\right\rangle + \left\langle\left(\frac{\partial u_1}{\partial x_3}\right)^2\right\rangle + \left\langle\left(\frac{\partial u_3}{\partial x_1}\right)^2\right\rangle + \left\langle\left(\frac{\partial u_2}{\partial x_3}\right)^2\right\rangle + \left\langle\left(\frac{\partial u_3}{\partial x_2}\right)^2\right\rangle\right] \\ & + 2\left[\left\langle\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\right\rangle + \left\langle\frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1}\right\rangle + \left\langle\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}\right\rangle\right]\Big\} \end{aligned} \tag{4}$$

From its definition as the curl of the velocity, the mean square vorticity can be determined as

$$\langle \omega_i \omega_i \rangle = \epsilon_{ijk}\epsilon_{imn} \left\langle \frac{\partial u_k}{\partial x_j}\frac{\partial u_n}{\partial x_m}\right\rangle \tag{5}$$

where ϵ_{ijk} is the alternating tensor. This can be expanded to yield

$$\begin{aligned} \langle \omega_i \omega_i \rangle \;=\; & \Big\{\left[\left\langle\left(\frac{\partial u_1}{\partial x_2}\right)^2\right\rangle + \left\langle\left(\frac{\partial u_2}{\partial x_1}\right)^2\right\rangle + \left\langle\left(\frac{\partial u_1}{\partial x_3}\right)^2\right\rangle + \left\langle\left(\frac{\partial u_3}{\partial x_1}\right)^2\right\rangle + \left\langle\left(\frac{\partial u_2}{\partial x_3}\right)^2\right\rangle + \left\langle\left(\frac{\partial u_3}{\partial x_2}\right)^2\right\rangle\right] \\ & - 2\left[\left\langle\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\right\rangle + \left\langle\frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1}\right\rangle + \left\langle\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}\right\rangle\right]\Big\} \end{aligned} \tag{6}$$

Thus twelve separate derivative correlations must be determined before the dissipation can be obtained, while only nine are required for the mean square vorticity. So imposing are these tasks that the former has never been accomplished by flow measurement, and the latter only recently and with great difficulty (v. Balint and Wallace 1985, Balint et al. 1987).

1.2 Isotropic Turbulence

The theory of isotropic turbulence is based on the fact that all the statistical measures of the flow must be invariant to reflections and rotations about all axes. The consequences of isotropy were first explored by Taylor (1935), and have been reviewed in detail by Batchelor (1953), Monin and Yaglom (1975), and Hinze (1975). The implications of isotropy for the dissipation and mean square vorticity are profound, and arise from the fact that all of the derivatives moments occurring in them are simply related. In particular,

$$\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle = \left\langle \left(\frac{\partial u_2}{\partial x_2}\right)^2 \right\rangle = \left\langle \left(\frac{\partial u_3}{\partial x_3}\right)^2 \right\rangle \tag{7}$$

$$\begin{aligned} 2\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle &= \left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle = \left\langle \left(\frac{\partial u_2}{\partial x_1}\right)^2 \right\rangle = \left\langle \left(\frac{\partial u_1}{\partial x_3}\right)^2 \right\rangle \\ &= \left\langle \left(\frac{\partial u_3}{\partial x_1}\right)^2 \right\rangle = \left\langle \left(\frac{\partial u_2}{\partial x_3}\right)^2 \right\rangle = \left\langle \left(\frac{\partial u_3}{\partial x_2}\right)^2 \right\rangle \end{aligned} \tag{8}$$

and

$$\left\langle \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} \right\rangle = \left\langle \frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1} \right\rangle = \left\langle \frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2} \right\rangle = -\frac{1}{2}\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle \tag{9}$$

Using these, the dissipation for isotropic turbulence reduces to

$$\varepsilon = 15\nu \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle \tag{10}$$

and the mean square vorticity to

$$\langle \omega_i \omega_i \rangle = 15 \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle \tag{11}$$

Thus, for isotropic turbulence, only a single mean square derivative need be obtained to completely determine both the mean square vorticity and the energy dissipation rate. (It will be seen below that the similarity of these two expressions is not unique to isotropic turbulence.)

1.3 Locally Isotropic Turbulence

There are few real turbulent flows in which the turbulence can be assumed to be isotropic. However, in high Reynolds number flows, the energy is mostly dissipated at the smallest scales of motion which do not receive energy directly from the mean flow, but through an energy transfer from large to small scales. Kolmogorov (1941) argued that this transfer process removes the directional information of the energy-containing eddies so that the small scales could be considered to be locally isotropic. For such locally isotropic flows, only the statistical properties of the smallest scales of motion would be expected to satisfy the isotropic relations.

Since in very high Reynolds number flows the mean square derivatives are largely determined by the smallest scales of motion, an important application of local isotropy has been to argue that the isotropic derivative relations given by equations 7 to 9 apply. Thus, in such flows, a single derivative measurement should be adequate to determine the dissipation and mean square vorticity.

Other consequences of local isotropy include the equi-partition of the dissipation between the three component kinetic energy equations, and the absence of a direct dissipation of the off-diagonal Reynolds stress components. These have been incorporated in most attempts to develop closure models for turbulence.

There is considerable evidence that local isotropy is not an adequate description of the velocity derivative moments for at least the finite Reynolds numbers associated with many turbulent laboratory flows. Browne et al. (1987) review a large number of turbulence derivative moment measurements and show that few satisfy the isotropic conditions of equations 7 — 9. Tables 1 and 2 (which have been adapted from their paper) summarize many of the results prior to 1987 for the derivative ratios defined by:

$$K_1 = 2\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle / \left\langle\left(\frac{\partial u_2}{\partial x_1}\right)^2\right\rangle, \qquad K_2 = 2\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle / \left\langle\left(\frac{\partial u_3}{\partial x_1}\right)^2\right\rangle$$

$$K_3 = 2\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle / \left\langle\left(\frac{\partial u_1}{\partial x_2}\right)^2\right\rangle, \qquad K_4 = 2\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle / \left\langle\left(\frac{\partial u_1}{\partial x_3}\right)^2\right\rangle$$

For isotropic (or locally isotropic) turbulence, all of these ratios should be unity. Clearly they are not, indicating that local isotropy is not an adequate description of the derivative relations for most of these flows.

In addition to the evidence for the lack of local isotropy in the velocity derivative moments, there is also evidence that the temperature derivative moments do not satisfy the conditions for local isotropy in a variety of turbulent flows of at least moderately high Reynolds numbers. For a locally isotropic scalar field, all three mean square derivatives should be equal. Thus the ratios defined by

$$M_1 = \left\langle\left(\frac{\partial \theta}{\partial x_1}\right)^2\right\rangle / \left\langle\left(\frac{\partial \theta}{\partial x_2}\right)^2\right\rangle, \qquad M_2 = \left\langle\left(\frac{\partial \theta}{\partial x_1}\right)^2\right\rangle / \left\langle\left(\frac{\partial \theta}{\partial x_3}\right)^2\right\rangle$$

should both be equal to unity for turbulence which is locally isotropic. None of the experiments summarized in Table 3 are consistent with the isotropic temperature derivative relations.

The purpose of this paper is to explore an alternative description of the turbulence; namely, *locally axisymmetric turbulence*. Local axisymmetry (with the 1-axis chosen as the axis of symmetry) will be seen to require that $K_1 = K_2$ and $K_3 = K_4$ for the velocity derivative ratios defined above with no restriction on their values. Thus, nearly all of the measurements listed in Tables 1 and 2 satisfy to within experimental error at least these conditions for local axisymmetry. For the locally axisymmetric temperature field the analogous requirement is that $M_1 = M_2$, again with no restrictions on the values. While not as convincing as Tables 1 and 2 (perhaps because of the presumed axis of symmetry), the results of Table 3 still indicate that local axisymmetry might be useful. After a review of the the constraints placed on the turbulence derivatives by homogeneity and axisymmetry, the concept of local axisymmetry will be defined and shown to provide a useful description of the turbulent fields considered.

2 Axisymmetric Homogeneous Turbulence

2.1 Homogeneous Turbulence

One of the primary objectives of this paper is to examine the derivative statistics of fields which are homogeneous, but not isotropic. Therefore it is important to examine what homogeneity (in the absence of further assumptions) implies about the derivative moments. The fundamental assumption of homogeneous turbulence is that the statistics of the field are independent of the location of the origin in space. Thus, the second order velocity correlation, Q_{ij}, defined by

$$Q_{ij}(\vec{r}) = \langle u_i(\vec{x})\, u_j(\vec{x}+\vec{r})\rangle \tag{12}$$

is a function only of the separation vector

$$\vec{r} = \vec{x'} - \vec{x} \tag{13}$$

From the definition of equation 12 and a simple translation of the spatial origin by $-\vec{r}$ it follows that for homogeneous turbulence,

$$Q_{ij}(\vec{r}) = Q_{ji}(-\vec{r}) \tag{14}$$

Homogeneity also has consequences for the velocity derivative moments. From the definitions it follows that

$$\begin{aligned} \left\langle \frac{\partial u_i}{\partial x_m}\frac{\partial u_j}{\partial x_n}\right\rangle &= \left\langle \frac{\partial u_i}{\partial x_m}\frac{\partial u'_j}{\partial x'_n}\right\rangle_{\vec{r}=0} \\ &= -\left.\frac{\partial^2 Q_{ij}}{\partial r_m \partial r_n}\right|_{\vec{r}=0} \end{aligned} \tag{15}$$

where $u'_i = u_i\left(\vec{x'}\right)$. By changing the order of differentiation it follows immediately that

$$\left\langle \frac{\partial u_i}{\partial x_m}\frac{\partial u_j}{\partial x_n}\right\rangle = \left\langle \frac{\partial u_i}{\partial x_n}\frac{\partial u_j}{\partial x_m}\right\rangle \tag{16}$$

Thus, of the eighty-one possible second-order derivative moments, only forty-five are independent.

If the incompressible continuity equation given by

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{17}$$

is multiplied by $\partial u_1/\partial x_1$, $\partial u_2/\partial x_2$ or $\partial u_3/\partial x_3$ and averaged, equation 16 can be used to transform the mixed derivative moments so that the following relationships result:

$$\left\langle \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\right\rangle + \left\langle \frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1}\right\rangle = -\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle \tag{18}$$

$$\left\langle \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\right\rangle + \left\langle \frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}\right\rangle = -\left\langle \left(\frac{\partial u_2}{\partial x_2}\right)^2\right\rangle \tag{19}$$

$$\left\langle \frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1}\right\rangle + \left\langle \frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}\right\rangle = -\left\langle \left(\frac{\partial u_3}{\partial x_3}\right)^2\right\rangle \tag{20}$$

These can be summed to yield

$$\left\langle \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\right\rangle + \left\langle \frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1}\right\rangle + \left\langle \frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}\right\rangle$$
$$= -\frac{1}{2}\left[\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle + \left\langle \left(\frac{\partial u_2}{\partial x_2}\right)^2\right\rangle + \left\langle \left(\frac{\partial u_3}{\partial x_3}\right)^2\right\rangle\right] \tag{21}$$

These relations were first derived by Taylor (1935).

Immediate consequences from substitution of equation 21 into equations 4 and 6 are that for homogeneous, incompressible flows,

$$\langle \omega_i \omega_i \rangle = 2\langle e_{ij} e_{ij} \rangle \tag{22}$$

and that

$$\varepsilon = \nu \left\langle \left(\frac{\partial u_i}{\partial x_j}\right)^2 \right\rangle \tag{23}$$

Thus for *homogeneous* turbulent flows, only nine derivative moments are needed to determine the dissipation *and* the mean square vorticity. It has been commonly assumed that equation 23 is valid only for isotropic turbulence , and it is often referred to as the "isotropic dissipation" (cf. Launder et al. 1975, Reynolds 1976, Taulbee 1988). Clearly, however, only the much less restrictive assumptions of homogeniety and incompressibility are required. [1]

For isotropic turbulence, the three cross derivative moments of equations 18 — 20 are equal as are the three derivatives on the right hand side, so that the isotropic relations of equation 9 follow immediately. The first two of these isotropic relations will be seen to hold for axisymmetric turbulence as well.

2.2 Axisymmetric Turbulence

A less restrictive hypothesis than isotropy is to assume only that all statistical measures have rotational symmetry about a given axis. This is the basic assumption behind the theory of axisymmetric homogeneous turbulence (hereafter referred to as simply axisymmetric turbulence) developed by Batchelor (1946) and Chandrasekhar (1950). For the evaluation of the dissipation and mean square vorticity which is of primary concern here, only the relations between the derivative moments are of interest. Somewhat surprisingly, they do not appear to have been derived before now. They can, however, be readily deduced from a Taylor expansion of the two-point velocity correlation near $\vec{r} = 0$ which is given by Chandrasekhar (1950) as

$$\begin{aligned} Q_{ij} &= 2(\alpha_{02} - \alpha_{22} + \beta_{02}) r_i r_j \\ &\quad + \left[-(2\alpha_{00} + \beta_{00}) + r^2\left\{(-4\alpha_{02} + 2\alpha_{22} - 3\beta_{02}) + \mu^2(2\beta_{02} - \beta_{22} - 8\alpha_{22})\right\}\right]\delta_{ij} \\ &\quad + \left[\beta_{00} + r^2\left(3\beta_{02} - 2\alpha_{22} + \beta_{22}\mu^2\right)\right]\lambda_i\lambda_j + 2r\mu(\lambda_i r_j + r_i\lambda_j)(2\alpha_{22} - \beta_{02}) \end{aligned} \tag{24}$$

[1] The equality of equations 1 and 23 can also be seen easily from the two forms of the incompressible kinetic energy equation, one using the mean square strain rate and the other the mean square deformation rate. For homogeneous flow, only the production terms remain to balance either form of the dissipation, thus they must be equal.

where $\vec{\lambda}$ is a unit vector in the preferred direction and $\mu = \vec{r} \cdot \vec{\lambda}/r$. The six coefficients — α_{00}, α_{02}, α_{22}, β_{00}, β_{02} and β_{22} — are independent invariants.

The two invariants α_{00} and β_{00} can be related to the components of the turbulence energy by

$$\alpha_{00} = -\frac{1}{2}\left\langle u_1^2 \right\rangle \tag{25}$$

and

$$\begin{aligned} \beta_{00} &= \left\langle u_1^2 \right\rangle - \left\langle u_2^2 \right\rangle \\ &= \left\langle u_1^2 \right\rangle - \left\langle u_3^2 \right\rangle \end{aligned} \tag{26}$$

These will not enter the derivative relations.

If the preferred direction is chosen to be the 1-direction, then $\vec{\lambda} = (1, 0, 0)$ and $\mu = r_1/r$. From equation 24 it is easy to see that for this orientation, the diagonal elements depend only on r_1^2, r_2^2 and r_3^2 ; i.e.

$$Q_{11} = -2\alpha_{00} - [2\alpha_{02} + 2\alpha_{22}]\, r_1^2 - 4\alpha_{02}\left(r_2^2 + r_3^2\right) \tag{27}$$

$$\begin{aligned} Q_{22} = {} & -(2\alpha_{00} + \beta_{00}) - [4\alpha_{02} + 6\alpha_{22} + \beta_{02} + \beta_{22}]\, r_1^2 \\ & - [2\alpha_{02} + \beta_{02}]\, r_2^2 - [4\alpha_{02} - 2\alpha_{22} + 3\beta_{02}]\, r_3^2 \end{aligned} \tag{28}$$

$$\begin{aligned} Q_{33} = {} & -(2\alpha_{00} + \beta_{00}) - [4\alpha_{02} + 6\alpha_{22} + \beta_{02} + \beta_{22}]\, r_1^2 \\ & - [4\alpha_{02} - 2\alpha_{22} + 3\beta_{02}]\, r_2^2 - [2\alpha_{02} + \beta_{02}]\, r_3^2 \end{aligned} \tag{29}$$

Note the similar forms of Q_{22} and Q_{33} (with the roles of r_2 and r_3 reversed) are consistent with the symmetry about the 1-axis.

For the off-diagonal terms, the condition for homogeneity of equation 14 implies that only three of the six components are independent. From equation 24 they can be derived as

$$Q_{12} = 2\,(\alpha_{02} + \alpha_{22})\, r_1 r_2 \tag{30}$$

$$Q_{13} = 2\,(\alpha_{02} + \alpha_{22})\, r_1 r_3 \tag{31}$$

$$Q_{23} = 2\,(\alpha_{02} - \alpha_{22} + \beta_{02})\, r_2 r_3 \tag{32}$$

The similarity of equations 30 and 31 is also a consequence of the symmetry about the 1– axis.

2.3 Derivative Relations for Axisymmetric Turbulence

It is straightforward to derive equations for the derivative moments in terms of the invariants α_{02}, α_{22}, β_{02} and β_{22}. Substitution of equation 24 into equation 15 yields the general relation

$$\begin{aligned} \left\langle \frac{\partial u_i}{\partial x_m}\frac{\partial u_j}{\partial x_n} \right\rangle = {} & (-2\alpha_{02} + 2\alpha_{22} - 2\beta_{02})\,(\delta_{in}\delta_{jm} + \delta_{im}\delta_{jn}) \\ & + \{(8\alpha_{02} - 4\alpha_{22} + 6\beta_{02})\,\delta_{mn} - (4\beta_{02} - 2\beta_{22} - 16\alpha_{22})\,\delta_{1m}\delta_{1n}\}\,\delta_{ij} \\ & + [(-6\beta_{02} + 4\alpha_{22})\,\delta_{mn} - 2\beta_{22}\delta_{1m}\delta_{1n}]\,\lambda_i\lambda_j \\ & - (4\alpha_{22} - 2\beta_{02})\,[\lambda_i\,(\delta_{1m}\delta_{jn} + \delta_{jm}\delta_{1n}) + \lambda_j\,(\delta_{1m}\delta_{in} + \delta_{im}\delta_{1n})] \end{aligned} \tag{33}$$

It can be shown from equation 33 that a consequence of axisymmetry is that all but fifteen of the forty-five independent derivative moments are zero. The non-zero moments include all of the mean square moments which enter the mean square strain rate and vorticity. If the 1-direction is chosen as the axis of symmetry they are given by

$$\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle = 4\alpha_{02} + 4\alpha_{22} \tag{34}$$

$$\left\langle\left(\frac{\partial u_2}{\partial x_2}\right)^2\right\rangle = \left\langle\left(\frac{\partial u_3}{\partial x_3}\right)^2\right\rangle = 4\alpha_{02} + 2\beta_{02} \tag{35}$$

$$\left\langle\left(\frac{\partial u_1}{\partial x_2}\right)^2\right\rangle = \left\langle\left(\frac{\partial u_1}{\partial x_3}\right)^2\right\rangle = 8\alpha_{02} \tag{36}$$

$$\left\langle\left(\frac{\partial u_2}{\partial x_1}\right)^2\right\rangle = \left\langle\left(\frac{\partial u_3}{\partial x_1}\right)^2\right\rangle = 8\alpha_{02} + 12\alpha_{22} + 2\beta_{02} + 2\beta_{22} \tag{37}$$

$$\left\langle\left(\frac{\partial u_2}{\partial x_3}\right)^2\right\rangle = \left\langle\left(\frac{\partial u_3}{\partial x_2}\right)^2\right\rangle = 8\alpha_{02} - 4\alpha_{22} + 6\beta_{02} \tag{38}$$

The cross-derivative moments can be derived immediately from these and the homogeneous constraints of equation 16 as

$$\left\langle\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\right\rangle = \left\langle\frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1}\right\rangle = -2\left(\alpha_{02} + \alpha_{22}\right) \tag{39}$$

$$\left\langle\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}\right\rangle = -2\left(\alpha_{02} - \alpha_{22} + \beta_{02}\right) \tag{40}$$

The invariants can be eliminated from equations 34 – 40 to yield relations between the derivatives which must be satisfied in axisymmetric turbulence. The results are

$$\left\langle\left(\frac{\partial u_1}{\partial x_2}\right)^2\right\rangle = \left\langle\left(\frac{\partial u_1}{\partial x_3}\right)^2\right\rangle \tag{41}$$

$$\left\langle\left(\frac{\partial u_2}{\partial x_1}\right)^2\right\rangle = \left\langle\left(\frac{\partial u_3}{\partial x_1}\right)^2\right\rangle \tag{42}$$

$$\left\langle\left(\frac{\partial u_2}{\partial x_2}\right)^2\right\rangle = \left\langle\left(\frac{\partial u_3}{\partial x_3}\right)^2\right\rangle \tag{43}$$

$$\left\langle\left(\frac{\partial u_2}{\partial x_3}\right)^2\right\rangle = \left\langle\left(\frac{\partial u_3}{\partial x_2}\right)^2\right\rangle \tag{44}$$

$$\left\langle\left(\frac{\partial u_2}{\partial x_2}\right)^2\right\rangle = \frac{1}{3}\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle + \frac{1}{3}\left\langle\left(\frac{\partial u_2}{\partial x_3}\right)^2\right\rangle \tag{45}$$

$$\left\langle\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}\right\rangle = \frac{1}{6}\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle - \frac{1}{3}\left\langle\left(\frac{\partial u_2}{\partial x_3}\right)^2\right\rangle \tag{46}$$

and

$$\left\langle\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\right\rangle = \left\langle\frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1}\right\rangle = -\frac{1}{2}\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle \tag{47}$$

Equations 34 – 40 can also be used to obtain alternative expressions for equations 45 – 46. Note that equation 47 is the same as for isotropic turbulence.

2.4 Determination of the Invariants

It is easy to see that a method for determing the axisymmetric invariants from directly measured quantities, other than from the complete correlation function itself, could be particularly useful. Since all of the derivative moments depend on only the four invariants — α_{02}, α_{22}, β_{02} and β_{22} — it is possible to uniquely determine the invariants in terms of a variety of quartets of derivative moments which are chosen to be independent. For example, if by some means $\left\langle (\partial u_1/\partial x_1)^2 \right\rangle$, $\left\langle (\partial u_1/\partial x_3)^2 \right\rangle$, $\left\langle (\partial u_2/\partial x_1)^2 \right\rangle$, and $\left\langle (\partial u_2/\partial x_3)^2 \right\rangle$ were determined, then the four invariants could be calculated from

$$\alpha_{02} = \frac{1}{8}\left\langle \left(\frac{\partial u_1}{\partial x_3}\right)^2 \right\rangle \tag{48}$$

$$\alpha_{22} = \frac{1}{4}\left[\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle - \frac{1}{2}\left\langle \left(\frac{\partial u_1}{\partial x_3}\right)^2 \right\rangle\right] \tag{49}$$

$$\beta_{02} = \frac{1}{6}\left[\left\langle \left(\frac{\partial u_2}{\partial x_3}\right)^2 \right\rangle + \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle - \frac{3}{2}\left\langle \left(\frac{\partial u_1}{\partial x_3}\right)^2 \right\rangle\right] \tag{50}$$

$$\beta_{22} = \frac{1}{2}\left[\left\langle \left(\frac{\partial u_2}{\partial x_1}\right)^2 \right\rangle + \left\langle \left(\frac{\partial u_1}{\partial x_3}\right)^2 \right\rangle - \frac{10}{3}\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle - \frac{1}{3}\left\langle \left(\frac{\partial u_2}{\partial x_3}\right)^2 \right\rangle\right] \tag{51}$$

This is, of course, only one of a number of possible combinations; it will be seen below, however, to be a particularly convenient one for the experimentalist.

2.5 The Dissipation and Mean Square Vorticity

The dissipation can be obtained by direct substitution of these relations into either equation 3 or equation 23. The result is

$$\varepsilon = \nu\left(60\alpha_{02} + 20\alpha_{22} + 20\beta_{02} + 4\beta_{22}\right) \tag{52}$$

The mean square vorticity can be obtained from equation 6 (or more directly by noting its equality with twice the mean square strain rate in equation 22) as

$$\langle \omega_i \omega_i \rangle = 60\alpha_{02} + 20\alpha_{22} + 20\beta_{02} + 4\beta_{22} \tag{53}$$

Thus both the dissipation and the mean square vorticity can be obtained once the four invariants α_{02}, α_{22}, β_{02} and β_{22} have been determined.

Of some interest (especially to turbulence modellers) are the components of dissipation which enter the equations for the individual components of the kinetic energy and Reynolds stress equations. For homogeneous turbulence, this "dissipation" tensor, D_{ik} is given by

$$D_{ik} = \nu\left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_j} \right\rangle \tag{54}$$

It is easy to show that the off-diagonal terms are zero for axisymmetric turbulence (as for isotropic turbulence). The component in the preferred direction is given by

$$\varepsilon_{||} = 20\,\alpha_{02} + 4\,\alpha_{22} \tag{55}$$

while the two perpendicular components are given by

$$\varepsilon_\perp = 20\,\alpha_{02} + 8\,\alpha_{22} + 10\,\beta_{02} + 2\,\beta_{22} \tag{56}$$

The components of the mean square vorticity can be similarly derived as

$$\left\langle {\omega_{||}}^2 \right\rangle = 20\,\alpha_{02} - 12\,\alpha_{22} + 16\,\beta_{02} \tag{57}$$

and

$$\left\langle {\omega_\perp}^2 \right\rangle = 20\,\alpha_{02} + 16\,\alpha_{22} + 2\,\beta_{02} + 2\,\beta_{22} \tag{58}$$

(Note that these two equations are incorrectly given by Chandrasekar 1950.)

The dissipation for axisymmetric turbulence can be expressed in terms of the velocity derivative moments, either by using equations 48 – 51 in equation 52 or by direct substitution of the derivative relations into the definition. For the choice of moments above, the result is

$$\varepsilon = \nu \left[\frac{5}{3} \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle + 2 \left\langle \left(\frac{\partial u_1}{\partial x_3}\right)^2 \right\rangle + 2 \left\langle \left(\frac{\partial u_2}{\partial x_1}\right)^2 \right\rangle + \frac{8}{3} \left\langle \left(\frac{\partial u_2}{\partial x_3}\right)^2 \right\rangle \right] \tag{59}$$

An alternative form depending only on measurements in the 1- and 2-directions is

$$\varepsilon = \nu \left[- \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle + 2 \left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle + 2 \left\langle \left(\frac{\partial u_2}{\partial x_1}\right)^2 \right\rangle + 8 \left\langle \left(\frac{\partial u_2}{\partial x_2}\right)^2 \right\rangle \right] \tag{60}$$

It is straightforward to derive relations for the individual components of vorticity as well as for the component dissipations. The exact form of these equations will depend on the choice of the preferred direction and the particular derivative moment combinations utilized to express the independent invariants.

3 Experimental Evidence for Local Axisymmetry

3.1 Testing for Local Axisymmetry

The results of the preceding section suggest a number of tests to which experimental determinations of the derivative moments can be subjected to determine whether or not they satisfy the constraints of locally axisymmetric turbulence. It will not be possible to establish sufficiency since it is usually impossible to measure all of the velocity derivatives. (This is, of course, no different than the requirements for establishing local isotropy.) However, it will be possible to show that an number of the necessary conditions for local axisymmetry given in equations 41 – 46 are satisfied. More importantly, it can easily be determined that a flow is *not* locally axisymmetric since only a single condition need be violated. It is obvious that if a flow is determined not to be locally axisymmetric, then it can not be locally isotropic. Of equal importance to showing that the non-zero derivative moments satisfy the appropriate relations is to establish that the remaining derivative moments are zero as required. When this much larger number of moments is considered (many of which can be measured), it will be seen that a strong inferential case for local axisymmetry can be made.

There are a number of experimental factors which complicate the determination of derivative moments in turbulent flows. One of the most important factors limiting derivative measurement is the spatial filtering arising from the finite length of the wires and from the finite separation of multiple wires. These have been analyzed in detail by Wyngaard (1968) for velocity measurement (using single and x-wires) and by Hussein and George (1990) for velocity difference measurement using parallel wire probes. In brief, these analyses show that all of the spatial dimensions of the probes must be on the order of or smaller than the Kolmogorov microscale.

Another significant problem arises from the need to determine derivatives in the flow direction by applying Taylor's frozen field hypothesis to time derivative measurements. Primary among several problems is the contamination of the estimated derivative by the fluctuating convection velocity inherent in turbulent flows. Lumley (1965), Wyngaard and Clifford (1977), and George et al. (1989) show that Taylor's hypothesis can be applied to derivative measurements without corrections only when the turbulence intensity is low (typically less than 10 %).

Two experiments will be considered in detail below – the round jet experment of Hussein and George (1989) (see also Hussein 1988) and the plane wake experiment of Browne et al. (1987). Both of these experiments (unlike many others) can be shown to have dealt adequately with the problems listed above. In particular, the hot-wires were sufficiently small to minimize the effects of spatial filtering, and the 'effective' turbulence intensity was low enough for the application of Taylor's hypothesis. The turbulence intensity in the far wake is naturally less than a few percent; however, for the round jet the *minimum* local intensity is greater than about 30%. Therefore the latter experiment utilized a whirling arm to superimpose a velocity on the probe to reduce the 'effective' velocity to less than 10%, and the contamination from the fluctuating convection velocity to less than a few percent.

A major source of error in both of the experiments arises from the difficulty in determining precisely the "effective" separation between the wires. This distance is needed to determine the x_2- and x_3- derivatives, and is particularly difficult to estimate when multiple x-wires are used. It is, in fact, the measurements made with these probe configurations which show the greatest scatter and lack of repeatability, and also present the greatest difficulties when evaluating the plane wake results. The jet experiment attempted to minimize these errors by using a special $A_X I$-probe consisting of only three wires (see Figure 1). Originally it was hoped that the upper and lower pairs of wires could be treated as independent x-wires, thereby providing two spatially separated estimates of both the streamwise and cross-stream velocities. Unfortunately, as recently pointed out by Hallbäck, Groth and Johansson (private communication), the $A_X I-$configuration cannot independently resolve the off-axis derivatives of the cross-stream velocities, so that the originally reported data for these derivatives were incorrect. It has been possible, however, to recover most of the information on the derivatives in queston by using the original data for both the parallel and $A_X I$-probes, and it is these revised data which will be discussed below.

3.2 The Turbulent Round Jet

The jet derivative moment measurements were made 70 exit diameters downstream of a turbulent round jet using a moving hot-wire probe. The exit Reynolds number of the jet was 100,000, and the value of R_λ at the measurement location was approximately 350. The mean centerline velocity at this location was 4.83 m/s and the probe velocity was 7.10 m/s. The derivatives with respect to the cross-stream directions were taken with the $A_X I$-probe, and with parallel wires (1-component of velocity only). A description of the facility, the measuring techniques and the original measurements has been given by Hussein (1988) and Hussein and George (1989). The above-mentioned problem with the $A_X I$-probe data was circumvented by recognizing that the net effect of differencing the two pairs of velocities was to produce the same velocity derivative that would have been generated by a single parallel wire operating at an angle to the x-direction, and with separation equal to twice the wire spacing. This is because when the cooling velocities for the individual wires are considered, the result of the subtraction is to remove the middle wire from consideration. If the flow is assumed to be locally homogeneous, the derivative measured by this probe is the average of the cross-stream derivatives of both the streamwise and cross-stream components of the velocity. Since the cross-stream derivative of the streamwise component was measured independently using parallel wires, the other component could be determined. While certainly not as desirable as the direct measurement originally intended, the results are a useful complement to the other derivative measurements which do not suffer from these difficulties, and without them no estimate of the invariants would have been possible.

The measurements of mean square derivatives which contribute to the dissipation are summarized in Figure 2. Also shown to facilitate comparison with the isotropic requirement is twice the value of $\left\langle (\partial u_1/\partial x_1)^2 \right\rangle$. While the striking departure from isotropy of the $\left\langle (\partial u_2/\partial x_3)^2 \right\rangle$ and $\left\langle (\partial u_3/\partial x_2)^2 \right\rangle$ moments may be questioned in view of their indirect determination,the non-isotropy of the remaining derivative moments is also clearly evident from the fact that different derivatives have different profiles, as well as from their failure to satisfy the isotropic relations (especially away from the jet centerline). All of the measured moments satisfy the axisymmetric relations of Section 2.3 to within the experimental errors which are estimated to be in the range of 10 – 20%. In particular,

$$
\begin{aligned}
\left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle &\approx \left\langle \left(\frac{\partial u_1}{\partial x_3}\right)^2 \right\rangle \\
\left\langle \left(\frac{\partial u_2}{\partial x_1}\right)^2 \right\rangle &\approx \left\langle \left(\frac{\partial u_3}{\partial x_1}\right)^2 \right\rangle \\
\left\langle \left(\frac{\partial u_2}{\partial x_3}\right)^2 \right\rangle &\approx \left\langle \left(\frac{\partial u_3}{\partial x_2}\right)^2 \right\rangle
\end{aligned}
$$

Figure 3 shows the profiles of the cross-correlation between the various derivative combinations available from the $A_X I$-probe, all of which should be zero in locally axisymmetric turbulence. All of these cross-moments are seen to be less by about an order of magnitude than the moments shown in Figure 2, the small residual values being consistent with the experimental errors and the expected slight departure from local homogeneity and local axisymmetry.

Figure 4 shows the invariants calculated from the measured derivative moments using equations 48 – 51. Especially apparent is the predominance of the third and fourth invariants, β_{02} and β_{22}. This is attributable to the relatively large values of $\left\langle (\partial u_3/\partial x_2)^2 \right\rangle$ and $\left\langle (\partial u_2/\partial x_3)^2 \right\rangle$ which do not affect the α_{02} and α_{22} invariants. In view of the manner in which these derivatives were obtained, further confirmation of these results should be obtained before too much is inferred from them.

Figure 5 shows the dissipation profile calculated from both the locally isotropic result of equation 10 and the locally axisymmetric result of equation 52 with the invariants obtained above. The isotropic result significantly underestimates the dissipation, at least relative to the locally axisymmetric estimate. It should be noted that the latter is significantly higher than the estimates obtained from the kinetic energy equation using the measured moments and ignoring the missing pressure-velocity correlations (Taulbee 1988). Whether the difference can be attributed entirely to these missing correlations or is due to measurement errors remains a subject for further investigation.

Figure 5 also shows the component dissipations from equations 55 and 56 together with one-third of the isotropic result. The large difference between $\varepsilon_\perp$ and the other two is almost entirely attributable to the large values of the two cross-stream derivatives determined from the $A_X I$-probe.

Profiles of the mean square vorticity components can easily be computed from the same invariants using equations 57 and 58, and are shown in Figure 6. In contra-distinction to the component dissipations, the streamwise component of the vorticity dominates the cross-stream component.

3.3 The Plane Wake

The plane wake mean square derivative results were obtained by Browne, Antonia and Shah (1987) in the far wake of a cylinder in a low turbulence wind tunnel. The measurements were taken at $x/d = 420$ using both parallel hot-wire and and pairs of x-wire probes. In all, nine of the twelve derivative moments entering the dissipation were measured, the missing moments being the cross-moments. The Reynolds number based on the mean flow of 6.7 m/s and the cylinder diameter of 2.67 cm was 1170, and R_λ was approximately equal to 20 at the measuring location. Thus the experiment provides a low Reynolds number counterpart to the relatively high Reynolds number jet experiment discussed above.

Figure 7 is adapted from Figure 4 of their paper, and shows the profiles of the measured derivative moments as a function of the y/L where L is the local velocity deficit half-width. Note that $2\left\langle(\partial u_1/\partial x_1)^2\right\rangle$ has also been plotted. It is immediately apparent that, as for the jet, local isotropy is not an acceptable description of this flow. Most striking from the figure are that to within the experimental error (estimated at 15 – 20 %)

$$\begin{aligned}\left\langle\left(\frac{\partial u_2}{\partial x_1}\right)^2\right\rangle &\approx \left\langle\left(\frac{\partial u_3}{\partial x_1}\right)^2\right\rangle \\ \left\langle\left(\frac{\partial u_1}{\partial x_2}\right)^2\right\rangle &\approx \left\langle\left(\frac{\partial u_1}{\partial x_3}\right)^2\right\rangle\end{aligned}$$

as required for local axisymmetry. Also, in spite of the fact that the profile shapes are somewhat different, the locally axisymmetric requirements that

$$\begin{aligned}\left\langle\left(\frac{\partial u_2}{\partial x_2}\right)^2\right\rangle &\approx \left\langle\left(\frac{\partial u_3}{\partial x_3}\right)^2\right\rangle \\ \left\langle\left(\frac{\partial u_3}{\partial x_2}\right)^2\right\rangle &\approx \left\langle\left(\frac{\partial u_2}{\partial x_3}\right)^2\right\rangle\end{aligned}$$

are satisfied for at least the former to within the experimental error. The relatively larger discrepancies in these pairs may be due to the difficulties in accurately and consistently measuring and maintaining the distances between the spatially separated x-wires used to obtain them. Whether the differences in profile shapes for the latter pair of moments is real or simply due to the limitations of the measurement techniques might be a subject for further investigation.

Figure 8 shows the dissipation calculated from the assumption of local axisymmetry using equations 59 and 60. Also shown is the isotropic estimate of equation 10 as well as the homogeneous estimate of equation 23 which uses all nine measured moments. [2] The agreement of the locally axisymmetric estimates with the locally homogeneous estimate is quite gratifying, especially since the former require far less experimental effort. Note that like the jet results above, the isotropic estimate is substantially different from the others.

Figure 9 shows the average wake invariants calculated using equations 34 – 36 and the two indicated sets of derivatives. Unlike the jet, the first invariant, α_{02}, is the largest but still is not dominant. Invariant estimates based on other derivative combinations show the same qualitative behavior, but differ in detail for all but α_{02}.

Figure 10 shows profiles of the dissipation and component dissipations calculated using the invariants of Figure 9. The dissipation differs slightly from the axisymmetric estimate of Figure 8 reflecting the different derivative moments used, but the difference is still within the experimental error. A major surprise in view of the strong anisotropy of the derivative moments is that the component dissipations are nearly equal to each other. Whether the lack of such equi-partition of dissipation in the jet is real or an artifact of the manner in which the data was processed should be a subject for further investigation.

Figure 11 shows the profiles of the vorticity components computed from the invariants. Like the jet, the streamwise component, $\overline{\omega_{\parallel}^2}$ is substantially larger than the cross-stream component, $\overline{\omega_{\perp}^2}$. This is in sharp contrast to the component dissipations and emphasizes the non-isotropy of this flow.

3.4 Discussion

Two experiments, a high Reynolds number round jet and a low Reynolds number plane wake, have been examined in detail. Of the combined total of 21 derivative moment measurements in the two flows, only the $\partial u_3/\partial x_2$ and the $\partial u_2/\partial x_3$ measurements in the wake fall outside the bounds of what can easily be attributed to experimental uncertainty. It is easy to show by writing out the derivative second moment transport equations that there can be no direct production of these moments from the interaction with the mean shear *unless* at least one of the cross-derivative moments does *not* satisfy the constraints of local axisymmetry (or local isotropy). It is well-known to turbulence modellers that such a direct production term must be included in the dissipation equation to successfully model turbulent shear flows (v. Launder et al. 1975). Thus the slight departures from local axisymmetry which have been noted may be a necessary feature of turbulent shear flows. It would appear then that a strong case for the near local axisymmetry of turbulent shear flows can be made, especially when the much greater number of results summarized in Tables 1 – 3 are considered.

[2] Browne et al. (1987) give only the isotropic estimate and several others based on approximations to equation 4, the most extensive of which yields a profile quite close to the homogeneous result used here.

Recently, Wallace and his co-workers (Balint et al. 1987, 1989) have presented measurements of the mean square vorticity and vorticity spectra in a turbulent boundary layer using a specially constructed nine-wire vorticity probe. Outside the immediate vicinity of the wall, the moments and spectra for the two cross-stream vorticity components were virtually identical, yet were distinctly different from those of the streamwise vorticity component. Similar results for the vorticity component spectra have also been obtained for the plane wake by Antonia et al. (1988). Thus, together with the measurements discussed above, there is strong evidence for local axisymmety of the turbulent vorticity field.

All of the flows considered have the common characteristic that the turbulence energy is primarily generated by the action of the Reynolds stresses on a single component of the turbulence. The velocity derivative data discussed herein are consistent with the assumption that it is this direction (the direction of the mean flow) which is the preferred *local* axis. While a different choice (for example, the principal axis of strain-rate) might be warranted for the temperature derivatives of Table 3, it does not seem appropriate for the velocity derivative data of most of the flows considered here, especially in view of the limited accuracy of the data itself. Why any of these choices should be appropriate can be a matter for debate (see below), but the success noted above may be simply a consequence of the fact that all of the flows considered are simple shear flows for which even the turbulence intensities are nearly axisymmetric. Alternatively, the exceptionally large values of several of the off-axis derivatives (if substantiated) might be giving a clue that it is the different character of the large scale structures for these flows which is responsible for the local axisymmetry. This possibility is particularly tantalizing in view of the fact that the flows differ in which of the derivatives are favored, and certainly in which coherent structures characterize them.

4 Summary and Conclusions

In the preceding sections, a theory of *locally axisymmetric turbulence* has been developed and shown to be reasonably consistent with a large body of experimental data. The theory provides an alternative to the assumption of local isotropy, and also provides additional credibility to those measurements which fail to confirm it. Antonia et al. (1986) summarize their search for local isotropy in turbulent shear flows as follows: " Isotropic relations between mean- square derivatives with respect to different spatial directions are satisfied by neither temperature nor velocity. To our knowledge, there is no firm indication that this departure from isotropy decreases with increasing Reynolds number." The results of this paper would appear to strongly confirm the first statement, and at least provide a basis for discussing the second.

While there is considerable evidence for local axisymmetry in turbulent flows, at least at laboratory Reynolds numbers, there can still be debate as to why and whether or not it will persist as the turbulence Reynolds number is increased. The conventional view of turbulence (Kolmogorov 1941)— that the turbulent fluctuations will tend to be more isotropic the smaller their scale — can be reconciled with the findings here if it is argued that it is only the larger scales which contribute the departures from isotropy. If so, then the anisotropy should disappear in the limit of very large Reynolds number. Browne et al. (1987), Sreenivasan et al. (1977), and Antonia et al. (1984) all cite evidence that it is in fact the disproportionate contributions to the low wavenumber spectra of the lateral and spanwise derivative which is responsible for the anisotropy. While this observation is not inconsistent with the conventional view, there is still no convincing evidence for these flows that these anisotropic contributions vanish with Reynolds number, nor is there confirmation that the high wavenumber spectra satisfy the isotropic spectral relations. In fact, the recent wind tunnel spectral measurements reported by Karyakin et al. (1990) to values of R_λ exceeding 3000 show persistent anisotropy throughout the inertial and viscous subranges. Both the derivative and spectral measurements are consistent with local axisymmetry, however.

George (1988) has suggested the alternative possibility that the small scale motions remain closely linked to the large scale coherent motions. If so, then anisotropy could be observed over the entire spectral range if the large scale motions are anisotropic. Thus, if the coherent structures of the motions are nearly axisymmetric because of the strong directional character of the mean flow from which they arise, then local axisymmetry might characterize the entire range of turbulent scales. This line of argument could account for the observed anisotropy of temperature gradients at very high Reynolds numbers, and also the recent results of Karyakin et al. cited above..

This paper has only set forth the beginnings of a theory of locally axisymmetric turbulence. In particular, it has focussed only on the second order derivative moments and the quantities computed from them. It is straightforward to use the results of Batchelor (1946) and Chandrasekar (1950) to explore the consequences of local axisymmetry for other statistical quantities. Of particular interest will be the relations governing the higher order derivatives, the third-order derivative moments, and the second and third order spectra since these enter the one and two point vorticity and dissipation balance equations. Experimental data for these quantities obtained over a range of Reynolds numbers will be essential to resolve the questions raised above about why local axisymmetry exists and what it implies about the nature of turbulence.

In conclusion, even though local axisymmetry is not as simple as local isotropy for either the theoretician or the experimentalist, it is considerably less complex than the alternative of full anisotropy or even local homogeneity alone. Perhaps most importantly, it places the dissipation, mean square vorticity, and their components within the reach of the experimentalist.

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List of Figures

Flow	Experimental Details		K_1	K_2	Reference
Quasi-homogeneous shear flow	$R_\lambda = 160$		1.30	1.30	Tavoularis & Corrsin 1981
Boundary layer	$R_d \approx 5x10^4$	$y/\delta \approx 0.08$	1.65	1.57	Verollet 1972
		≈ 0.32	1.72	1.59	
Pipe	$R_d = 5x10^4$	Near wall	3.3	1.8	Laufer 1954
		Half radius	1.0	0.83	
	$= 5x10^5$	Near wall	1.73	1.73	
		Half radius	1.40	1.12	
	$= 9x10^4$	Near wall	1.22	1.22	Lawn 1971
		Half radius	1.27	1.27	
Circular jet	$R_d = 10^5$	$r/x = 0$	0.98	0.98	Wygnanski & Fiedler 1969
		$= 0.05$	1.25	1.25	
		$= 0.1$	1.77	1.77	
Round Plume	$R_\lambda \approx 125$	Centerline	2.2	—	Beuther 1980
		$r/x = 0.05$	1.8	—	
		$= 0.09$	1.6	—	
Plane jet (without external flow)	$R_\lambda = 990$	$y/x = 0$	1.47	1.47	Gutmark & Wygnanski 1976
		$= 0.05$	1.59	1.59	
		$= 0.1$	1.77	1.77	
	$R_\lambda = 204$	Centerline	1.32	1.32	Antonia et al. 1982
	$R_\lambda = 300$	Centerline	1.33	1.33	Everitt & Robins 1978
	$= 200$	Halfwidth	2.0	2.0	
Plane jet (with external flow)	Strong jet	Centerline	1.82	1.82	Everitt & Robins 1978
		Halfwidth	2.0	2.0	
	Weak jet	Centerline	1.18	1.54	
		Halfwidth	1.18	1.33	
Mixing layer	$R_\lambda = 330$		1.74	—	Champagne et al. 1976
Two-dimensional cylinder wake	$R_d = 2.7x10^3$	$y/d = 0$	0.99	1.23	Fabris 1974
		$= 6$	1.15	1.25	
	$R_d = 1.905x10^4$	$y/d = 0$	1.16	1.12	Champagne 1978

Table 1: Velocity derivative ratios K_1 and K_2

Flow	Experimental details		K_3 †	K_4 †	Reference
Boundary layer	$R_\delta = 5x10^4$	$y/\delta \approx 0.08$	0.76	0.8	Verollet 1972
		≈ 0.32	0.76	0.78	
Pipe	$R_d = 5x10^4$	Near wall	0.40	0.40	Laufer 1954
		Half radius	0.69	0.56	
	$R_d = 5x10^5$	Near wall	0.93	0.93	
		Half radius	0.80	0.80	
	$R_d = 9x10^4$	Across pipe	1.0	—	Lawn 1971
Circular jet	$R_d = 1.0x10^5$	r/x = 0	1.2	1.2	Wygnanski & Fiedler 1969
		= 0.05	0.83	0.83	
		= 0.1	0.32	0.32	
Plane jet (without external flow)	$R_\lambda = 204$	Centerline	0.57	—	Antonia et al. 1984

† The cylindrical coordinate equivalent is used for pipe flows.

Table 2: Velocity derivative ratios K_3 and K_4

Flow	Experimental details		M_1	M_2	Reference
Homogeneous Shear Flow	$R_\lambda = 160$		1.18	1.18	Tavoularis & Corrsin 1981
Boundary layer	$R_\theta = 5730$	$y/\delta \approx 0.06$	1.28	1.68	Sreenivasan et al. 1977
		≈ 0.12	1.14	1.30	
		≈ 0.18	1.18	1.38	
		≈ 0.24	1.15	1.47	
	$R_\theta = 2000$	$y^+ \approx 25$	2.7	4.0	Krishnamoorthy & Antonia 1987
		≈ 50	2.0	2.0	
		≈ 100	1.4	1.6	
Plane wake	$R_\lambda = 36$	Centerline	1.9	1.7	Antonia & Browne 1986
		$y/L \approx 1$	1.9	1.9	
		$y/L \approx 1.4$	3.1	2.3	
		$y/L \approx 2$	2.3	1.6	
Plane jet	$R_\lambda \approx 160$	Centerline	2.0		Antonia et al. 1986

Table 3: Temperature dervivative ratios M_1 and M_2