1

THE ZERO PRESSURE-GRADIENT TURBULENT BOUNDARY LAYER

William K. George and Luciano Castillo
State University of New York at Buffalo
Buffalo, NY 14260

May 1, 1997

Abstract

A theory for the zero pressure-gradient turbulent boundary layer is proposed based on a similarity analysis of the boundary layer equations. The *Asymptotic Invariance Principle (AIP)* requires that properly scaled profiles reduce to similarity solutions of the inner and outer equations separately in the limit of infinite Reynolds number. The inner profiles scale with ν and u_* as in the classical analysis. The outer flow, however, requires two velocity scales, u_* and U_∞, in addition to δ, the boundary layer thickness. The velocity deficit law, in particular, scales with U_∞ instead of u_* as in the classical analysis.

The fact that both inner and outer scaled profiles describe the entire boundary layer for finite values of δ^+, but reduce to inner and outer profiles in the limit, is used to determine their functional forms in the "overlap" region which both retain. Since the ratio of scales, u_*/U_∞, is Reynolds number dependent, so must be the "overlap" region. To leading order the profiles in this overlap region are power laws, as is the friction law. The three parameters in these power laws depend on δ^+ (or R_θ), but are asymptotically constant and are linked by a constraint equation.

Application of the AIP to the Reynolds stress equations is used to determine the proper scaling laws for many of the turbulence quantities in the inner and outer layers. A surprising result is that in the outer layer (and overlap region) some of them scale with both u_* and U_∞, hence they are never Reynolds number independent, except in the limit. Other consequences of the theory are that the asymptotic values of δ_*/δ and θ/δ are non-zero constants. By considering when the Reynolds stress and energy dissipation become Reynolds number independent, it is possible to estimate that the asymptotes for c_f, $d\theta/dx$, and γ should not be reached until $R_\theta > 10^5$, which is beyond the range of existing experiments.

Finally, it is suggested that there exists a *mesolayer* in the region approximately defined by $10 < y^+ < 300$ in which the energy dissipation evolves from $\nu q^2/y^2$ to q^3/y. Thus the Reynolds stress and mean flow equations retain a Reynolds number dependence, even though the terms explicitly containing the viscosity are negligible. A simple turbulence model suggests that a term proportional to ${y^+}^{-1}$ should be added to the overlap velocity profile to account for this effect. Because of the mesolayer, the overlap solution is clearly evident only beyond $y^+ > 300$, well outside where it has commonly been sought.

The abundant experimental data are carefully examined and shown to be in reasonable agreement with the new theory. Further it is shown that all of the data can be described with just three constants and a semi-empirical equation for the variation of γ with δ^+.

*This paper was presented at the Reynolds Symposium on Turbulence, Asilomar, CA in April 1993. It is a revised and expanded version of the paper by George et al. 1992 at the 13^{th} Symposium on Turbulence, Rolla, Mo. Sept. 1992.

Contents

1 Prologue

This is a paper about the zero-pressure gradient turbulent boundary layer, and aside from a brief appendix on channel flow, it is only about the zero-pressure gradient boundary layer. There are a number of other wall-bounded flows that are at least as interesting: pressure-gradient boundary layers, pipe and channel flows, and wall jets, to name but a few. But the esssential difficulty of the subject is contained in the zero pressure gradient boundary layer, and any general principles which apply to it should be applicable to all — if they are in fact principles and general.

This paper is a review, but not in the traditional sense. It does not attempt to exhaustively list the huge body of work on even the zero pressure gradient boundary layer. To attempt to do so would be redundant in view of the excellent reviews of the past few years which have done exactly that (cf. Sreenivasan 1989, Gad-el-Hak and Bandyopadhyay 1994 and Smits and Dussauge 1996). Nor does this paper focus on the inadequacies of the data or the classical theories; these have been exhaustively treated in the same references and others. Instead this paper establishes a methodology for evaluation of theory and experiment which is based entirely on the averaged Navier-Stokes equations. No empirical scaling laws are proposed, but some general principles for deriving them are set forth. Application of them provides both the context for review of old ideas and for the evolution of new ones. The result of this critique and innovation is a comprehensive theory for the zero pressure gradient turbulent boundary layer, the methodology for which can be easily extended to other flow situations.

2 Introduction

There are few problems in turbulence which have been more generally regarded as solved than the scaling laws for the zero-pressure gradient boundary layer. The analysis of channel and pipe flow by Millikan

(1938) which matched inner and outer scaling laws (the Law of the Wall and the Velocity Deficit Law) to obtain logarithmic velocity and friction laws is widely considered to be classical. Similarly, the extensions by Clauser (1954), Hama (1954), Coles (1962) and others of Millikan's arguments to boundary layers with pressure gradients, roughness and compressibility have been accepted virtually without question for the past four decades. An important reason for this has been the apparent agreement of experimental data with the theoretical results. On the other hand, perhaps the editor's note to the paper of Long and Chen (1981) gives a clue that dissent has simply been suppressed by not publishing it.

There are a number of features of the Millikan/Clauser theory which, if not unsatisfying, are at least interesting. Among them:

- (i) The velocity profile disappears in the limit of infinite Reynolds number (i.e., $U/U_\infty = 1$);
- (ii) The outer length scale is not proportional to any integral length scale, and in fact blows up relative to them as the Reynolds number becomes infinite (i.e., δ/δ_* and $\delta/\theta \to \infty$); and
- (iii) The shape factor approaches unity in the infinite Reynolds number limit (i.e., $H = \delta_*/\theta \to 1$).

While these might be considered plausible if the limit is approached by increasing the free stream velocity or by decreasing the viscosity, they seem less reasonable if the limit is approached by simply increasing the streamwise distance (i.e., by proceeding along the plate). In addition, a practical objection can be raised because the logarithmic forms permit only empirical models for the streamwise development of the boundary layer parameters.

It might be argued that some free shear flows (like wakes) share these characteristics; however, this is of small comfort since the very essence of a boundary layer is the continuing loss of momentum to the wall. Experimentally there are also reasons for concern, since no shape factors below about 1.16 have been reported, and boundary layer profiles seem to collapse as well with momentum and displacement thicknesses as with the boundary layer thickness determined from the profile (e.g., $\delta_{0.99}$). Nonetheless it has somehow been possible to live with these 'problems'. Clauser (1954) and Tennekes and Lumley (1972), for example, use the displacement thickness as the outer length scale in the analysis of equilibrium boundary layers, even though its use is inconsistent with the Millikan/Clauser analysis (see *(ii)* above). Acceptance of these ambiguities can in part be understood because of the relatively limited range of Reynolds numbers at which experiments have been performed, but in larger part it probably should be attributed to the absence of rational alternative theories. This is somewhat surprising since an even more fundamental objection can be raised to the classical theory: It begins with two scaling 'laws', only one of which (the Law of the Wall) is derivable from the equations of motion; the other (the Velocity Deficit Law) is empirical.

This paper reconsiders first the theoretical foundations of the laws on which the Clauser/Millikan analysis is based, and shows one of them (the velocity deficit law) to be inconsistent with similarity of the outer equations of motion. Then it examines the consequences of an alternative formulation of the outer scaling law which *is* consistent with similarity in the limit of infinite Reynolds number. When combined with the Law of the Wall, this new deficit law is shown to lead to velocity and friction laws which are power laws. The new theory removes some of the troubling aspects of the earlier theory; in particular, the outer length scale can be identified with either the momentum or the displacement thicknesses, and the asymptotic shape factor is greater than unity. The theory proposed here can be extended to boundary layers with pressure gradient, surface roughness, compressibility, thermal effects and buoyancy; these will be reported elsewhere.

Part I
Theoretical Considerations

3 Governing Equations and Boundary Conditions

The equation of motion and boundary conditions appropriate to a zero pressure gradient turbulent boundary layer (with constant properties) at high Reynolds number are well-known to be given by (Tennekes and Lumley 1972)

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\frac{\partial}{\partial y}\left[-<uv>+\nu\frac{\partial U}{\partial y}\right]-\left\{\frac{\partial}{\partial x}\left[<u^2>-<v^2>\right]\right\} \quad (1)$$

where $U\rightarrow U_\infty$ as $y\rightarrow\infty$ and $U=0$ at $y=0$. The $<v^2>$ term arises from the substitution for the pressure from the integral of the y-momentum equation (v. Tennekes and Lumley 1972). Both of the terms in curly brackets are of second order in the turbulence intensity and are usually neglected; they are kept here since, unlike the other neglected terms, they do not become vanishingly small as the Reynolds number increases.

The presence of the no-slip condition precludes the possibility of similarity solutions for the entire boundary layer. If the viscous term is simply neglected, the solutions lose the ability to satisfy the no-slip condition, so there is no boundary layer and no drag — which is precisely what classical laminar boundary theory was invented by Prandtl to avoid. Thus the zero pressure gradient boundary layer is another classical singular perturbation problem where an inner length scale must be defined to insure that the viscous term survives near the wall, even in the limit of infinite Reynolds number — in effect a boundary layer inside of a boundary layer (cf. Tennekes and Lumley 1972). The result is that the turbulent boundary layer itself is governed by two distinct regions: An outer region comprising most of the boundary layer where the single point Reynolds-averaged equations are effectively inviscid; and an inner region very close to the wall where only the viscous term is dominant.

So solutions are sought which asymptotically (at infinite Reynolds number) satisfy the following outer and inner equations and boundary conditions:

- Outer Region

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\frac{\partial}{\partial y}\left[-<uv>\right]-\left\{\frac{\partial}{\partial x}\left[<u^2>-<v^2>\right]\right\} \quad (2)$$

  where $U\rightarrow U_\infty$ as $y\rightarrow\infty$.

- Inner (or near wall) region

$$0=\frac{\partial}{\partial y}\left[-<uv>+\nu\frac{\partial U}{\partial y}\right] \quad (3)$$

  where $U=0$ at $y=0$.

The neglected terms in both inner and outer equations vanish identically only at infinite Reynolds number (v. Tennekes and Lumley 1972). [1] However, there is nothing in the development of these equations which

[1] This can most easily be shown *a posteriori* by substituting the scaled solutions into the full equations, then using the friction law to get the asymptotic dependence.

precludes their approximate validity from the time the flow begins to develop unsteady disturbances, as long as the Reynolds number is large.

Equation 3 for the inner region can be integrated directly to obtain

$$-<uv>+\nu\frac{\partial U}{\partial y}=\frac{\tau_W}{\rho}\equiv u_*^2 \tag{4}$$

where τ_W is the wall shear stress and u_* is the corresponding friction velocity defined from it. It is clear that in the limit of infinite Reynolds number, the total stress is constant across the inner layer (but only in this limit), and hence its name "the Constant Stress Layer". It should be noted that the appearance of u_* in equation 4 does not imply that the wall shear stress is an independent parameter (like ν or U_∞). It enters the problem only because it measures the forcing of the inner flow by the outer; or alternatively, it can be viewed as measuring the retarding effect of the inner flow on the outer. Thus u_* is a *dependent* parameter which must be determined by matching solutions of the inner and outer equations.

It is also interesting to note that the inner layer occurs only because of the necessity of including viscosity in the problem so that the no-slip condition can be met. The outer layer, on the other hand, is dominated by inertia and the effects of viscosity enter only through the matching to the inner layer. Thus the outer flow is effectively governed by inviscid equations, *but with viscous-dominated inner boundary conditions set by the inner layer.*

4 The Velocity Scaling Laws

It has been customary to seek solutions to the governing equations which depend only on the streamwise coordinate through a local length scale $\delta(x)$, v. Monin and Yaglom 1971. The only parameters arising in the governing equations and boundary conditions are the free stream velocity, U_∞, the kinematic viscosity,ν, and the friction velocity, u_*. Of these, the latter is clearly a dependent parameter, and hence determined by the rest; i.e., $u_* = u_*(U_\infty, \delta, \nu)$. From the Buckingham Pi Theorem and dimensional considerations alone it follows immediately that there are only two independent dimensionless ratios. Convenient choices are u_*/U_∞ and $u_*\delta/\nu$, so the functional dependence of the former on the latter can be written as

$$\frac{u_*}{U_\infty}=\sqrt{\frac{c_f}{2}}=g\left(\delta^+\right) \tag{5}$$

where

$$\delta^+\equiv\frac{u_*\delta}{\nu}=\delta^+ \tag{6}$$

and

$$c_f\equiv\frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}=2\frac{u_*^2}{U_\infty^2} \tag{7}$$

Note that a consequence of equation 5 is that either (but not both!) u_* or U_∞ can be used be for scaling as long as δ^+ is retained and is finite.

The limiting value of u_*/U_∞ will be seen to of considerable interest, both in its own right, and because it determines if $\delta^+\to\infty$ and $U_\infty\delta/\nu\to\infty$ as $U_\infty x/\nu\to\infty$. For the moment, it will be *assumed* that this is the case, so that any of the limits can be used interchangeably. It will be possible to show*a posteriori* that this assumption is consistent with the derived friction law, the momentum integral equation, and a boundary layer which continues to grow while the skin friction becomes vanishingly small.

Since there are no other independent parameters, the local mean velocity profile must described by $U = U(y, \delta, U_\infty, \nu)$. Application of the Buckingham Pi theorem and using equation 5 yields a number of possibilities, *all of which describe the variation of the velocity across the entire boundary layer.* Among them are:

$$\frac{U}{u_*} = f_i\left(\frac{yu_*}{\nu}, \delta^+\right) \tag{8}$$

$$\frac{U - U_\infty}{U_\infty} = f_o\left(\frac{y}{\delta}, \delta^+\right) \tag{9}$$

$$\frac{U - U_\infty}{u_*} = F_o\left(\frac{y}{\delta}, \delta^+\right) \tag{10}$$

2

It is of particular interest to investigate the behavior of the scaled profiles as δ^+ becomes large, or other Reynolds numbers for that matter (e.g., $U_\infty \delta/\nu$ or $U_\infty x/\nu$). Thus in the limit as $\delta^+ \to \infty$, the only dimensionless profiles of interest will be those which produce finite values of the scaled velocity (the left hand side) for finite values of the scaled distance from the wall (the remaining argument on the right hand side). For example, if equation 8 becomes asymptotically independent of δ^+, it loses its only link to free stream velocity, and can at most describe a limited region very close to the wall, i.e.,

$$\frac{U}{u_*} = f_{i\infty}\left[\frac{yu_*}{\nu}\right] \tag{11}$$

This is, of course, the familiar *Law of the Wall* expressed in inner variables as originally proposed by Prandtl (1932).

A similar limiting argument for f_o and F_o yields two quite different candidates for an outer profile; namely,

$$\frac{U - U_\infty}{U_\infty} = f_{o\infty}\left(\frac{y}{\delta}\right) \tag{12}$$

and

$$\frac{U - U_\infty}{u_*} = F_{o\infty}\left(\frac{y}{\delta}\right) \tag{13}$$

Both cannot, of course, be Reynolds number independent (and finite) in the limit if the ratio u_*/U_∞ continues to vary (as the Millikan/Clauser theory requires, see below).

The first form given by equation 12 has only been fleetingly considered by the fluid dynamics community, and discarded in favor of the second alternative. Millikan, for example, appears to have considered it briefly, noted that it leads to self-preserving power law solutions of the outer equations, and then dismissed these solutions as interpolation formulas. Clauser (1954) (see also Hinze 1975) plotted only the highest and lowest Reynolds number data of Schultz-Grunow (1941) in deficit form and concluded that the collapse

[2] Note that since u_*/U_∞ and δ^+ (or $u_*\delta/\nu$) are related by equation 5, only the later need be retained in equations 8 – 10. This fact seems to have escaped Monin and Yaglom 1971 who dismiss a separate dependence on u_*/U_∞, only on empirical grounds.

using equation 12 was not as satisfactory as that obtained using the deficit form of equation 13. There is no evidence that either of these conclusions has been refuted, or even questioned (even though there were numerous other Schultz-Grunow profiles at intermediate Reynolds numbers which showed a clear trend with increasing Reynolds number toward the highest).

The second form given by equation 13 is the traditional choice. It was originally used by Stanton and Pannell (1914) for pipe flows and adapted by von Karman (1930) to the boundary layer. Clauser (1954), following Millikan (1938), matched the 'inner' scaling of equation 11 to the 'outer' scaling of equation 13 in the limit of infinite Reynolds number to obtain the familiar *inertial sublayer* profiles as

$$\frac{U}{u_*} = \frac{1}{\kappa} ln \left(\frac{y}{\eta}\right) + B_i \tag{14}$$

$$\frac{U - U_\infty}{u_*} = \frac{1}{\kappa} ln \left(\frac{y}{\delta}\right) + B_o \tag{15}$$

and a friction law given by

$$\frac{U_\infty}{u_*} = \sqrt{\frac{2}{c_f}} = \frac{1}{\kappa} ln \delta^+ + (B_i - B_o) \tag{16}$$

where κ, B_i, and B_o are presumed to be universal constants. (Actually Millikan only analyzed the pipe and channel flows, but indicated that a boundary layer application would follow. One can presume that this second paper was the 1954 Clauser paper.) Note that in the limit as $\delta^+ \to \infty$, equation 16 requires that $u_*/U_\infty \to 0$ for finite values of the constants.

By substituting the inner and outer scaling laws into the defining integrals for the displacement and momentum thicknesses, Clauser (1954) showed that

$$\frac{\delta_*}{\delta} = A_1 \frac{u_*}{U_\infty} \tag{17}$$

$$\frac{\theta}{\delta} = A_1 \frac{u_*}{U_\infty} \left[1 - A_2 \frac{u_*}{U_\infty}\right] \tag{18}$$

where A_1 and A_2 should be universal constants which can be evaluated from integrals of the outer velocity profile function. It follows that the shape factor is given by the asymptotic relation

$$H = \frac{\delta_*}{\theta} = [1 - A_2 \frac{u_*}{U_\infty}]^{-1} \tag{19}$$

Thus as $\delta^+ \to \infty$ and $u_*/U_\infty \to 0$ (if it in fact does), $H \to 1$.

The underlying assumption of the above matching is that the inner and outer scaling laws used for the profiles, in fact, have a region of common validity (or overlap) in the limit as $\delta^+ \to \infty$. Long and Chen (1981) have remarked that it is indeed strange that the matched layer between one characterized by inertia and another characterized by viscosity does not depend on both inertia and viscosity, but only inertia (hence the term 'inertial sublayer', Tennekes and Lumley 1972). They further suggested that this might be a consequence of improperly matching two layers which did not overlap. The fact that the limiting ratio of the outer length scale δ to both of the commonly used integral length scales, δ_* and θ, is infinite lends considerable weight to their concern. In particular, this implies that from the perspective of the outer flow, the boundary layer does not exist at all in the limit of infinite Reynolds number. If one imagines approaching this limit along a semi-infinite plate where the boundary layer continues to grow, the outer length scale increases faster than any dynamically significant integral length. This is particularly troubling since δ itself is unspecified by the theory and can not be related to physically measurable length scales except through the degenerate expressions above.

Theoretical objections notwithstanding, there has been widespread acceptance (canonization might be more accurate) of the Millikan/Clauser theory because it is believed to be consistent with the experimental data. This has never been entirely true, and as better data have been acquired it has become even more evident to be false. Even Coles (1956), whose careful determinations of the constants are the most often cited, expressed puzzlement at the apparent failure of the outer velocity profile (the "wake" constant in particular) to achieve Reynolds number independence when scaled with u_*. The recent careful review of Gad-el-Hak and Bandyopadyhay (1994) lists a number of experiments where persistent Reynolds number trends in the mean profile deficit are observed, even at relatively high Reynolds number. And there have been nagging problems in trying to reconcile direct measurements of wall shear stress (either by sensors on the wall or direct measurements in the linear layer) with the shear stress inferred from the logarithmic region using the "accepted" constants, the so-called Clauser method (v. Section 22 for a detailed discussion). Finally, when the same scaling arguments are extended to the higher order turbulence moments (second moments and above), they fail to collapse the data outside of the viscous sublayer ($y^+ > 10$ or so) (v. Gad-el-Hak and Bandyopadyhay 1994). In fact, when applied to quantities like the temperature field in the forced convection boundary layer, they seem to fail completely (v. Bradshaw and Huang 1995).

In view of the above, it is useful to examine whether u_* should be a scaling parameter for the outer flow at all, especially since there is an alternative which does not use it. First it should be noted that it is only in the limit of infinite Reynolds number where the inner layer is truly a constant stress layer. Thus, only in this limit is the shear stress experienced by the outer flow exactly that measured by u_*^2 at the wall. At all finite Reynolds numbers it only approximately measures the effect of the inner layer on the outer. While the use of u_* as an outer scaling parameter may give reasonable results over a rather large range of Reynolds numbers, it can not be an appropriate choice for the cornerstone of an asymptotic analysis of the outer boundary layer. (This has also been pointed out by Panton 1990 who tries to "fix" the problem with a higher order analysis while retaining the same deficit law.) This is in contrast with fully-developed turbulent pipe or channel flow where the overall balance between pressure and viscous forces on a section of the flow dictates that both the inner and outer flow scale with u_*, a direct consequence of the streamwise homogeneity (v. Appendix I for a complete derivation). An obvious inference from these observations is that the wall layers of these homogeneous flows might be fundamentally different from those of the inhomogeneous boundary layer, contrary to popular belief (cf. Monin and Yaglom 1972, Tennekes and Lumley 1972).

There have been numerous attempts to place the Millikan/Clauser theory on a more secure footing and extend it to higher order, especially notable among them: Bush and Fendell 1972, Long and Chen 1981, and Panton 1990. All began with the same velocity deficit, and therefore will not be considered further here. In the remainder of this paper, the alternative formulation of the outer profile given by equation 12 and the Law of the Wall will be shown to follow directly from the hypothesis that the outer and inner flow equations should admit to similarity solutions of the more general type described by George (1989, 1995). The consequences of matching two regions governed by different velocity scales will be explored, and the governing relations for a variety of turbulence quantities will be derived. The latter will be seen to give insight into when similarity should be expected in real boundary layers. Finally, the new theory will be compared to the experimental data.

5 The Asymptotic Invariance Principle

For reasons that will become clear below, the traditional approach to the boundary layer equations has been to abandon the possibility of full similarity, and seek instead local similarity solutions of the type described earlier. As a consequence, the local similarity solutions obtained were not necessarily similarity solutions of equations 1, 2 or 3 since no effort was made to insure that they were. Instead investigators attempted to establish their validity by experiment alone, dismissing as unimportant terms in the mean momentum equations which were inconsistent and ignoring the higher order moment equations altogether.

An alternative approach (which does not seem to have been previously attempted) is to seek full similarity solutions of the inner and outer equations separately. Since these equations (equations 2 and 3) are themselves exactly valid only in the limit of infinite Reynolds number, then their full similarity solutions will also be exactly valid only in this limit. Seen another way, since the equations themselves have neglected terms which are Reynolds number dependent and lose these terms only in the infinite Reynolds number limit; solutions to these full equations will likewise be Reynolds number dependent and lose this dependence only at infinite Reynolds number. This idea will be referred to as the *Asymptotic Invariance Principle.* (This term appears to have first been used by Knecht 1990, but with a slightly different meaning.)

The *Asymptotic Invariance Principle* can be applied to turbulent free shear flows, as well as boundary layer flows. Similarity solutions for free shear flows (when they exist) are, in fact, infinite Reynolds number solutions because the equations from which they are derived are strictly valid only at infinite Reynolds number (cf. George 1989). The difference in application here is that for the boundary layer there will be two different scaling laws to be applied to the complete solution — one which reduces to a full similarity solution of the *outer* equations at infinite Reynolds number, and another which reduces to a full similarity solution of the *inner* equations in the same limit. For finite Reynolds numbers, the Reynolds number dependence of the equations themselves, however weak, dictates that the solutions can not be similarity solutions anywhere. But, as noted above, this is no different than for free shear flows which only asymptotically show Reynolds number independence.

In the following sections, the Asymptotic Invariance Principle will be applied to some of the single point equations governing the zero pressure gradient turbulent boundary layer. In particular, solutions will be sought which reduce to full similarity solutions of the equations in the limit of infinite Reynolds number, first for the inner equations and then for the outer. The form of these solutions will determine the appropriate scaling laws for *finite* as well as infinite Reynolds number, since alternative scaling laws could not be independent of the Reynolds number in the limit. Once the method has been established by application to the equations governing the mean momentum, then the same principle will be applied to equations governing the Reynolds stress equations and the statistical quantities appearing in them. There is, of course, no reason why the Asymptotic Invariance Principle can not be applied to equations governing any statistical quantity, including multi-point equations, and some inferences will be made as to what the results of such application might be.

6 Full Similarity of the Inner Equations

In keeping with the Asymptotic Invariance Principle set forth above, solutions are sought which reduce to similarity solutions of the inner equations and boundary conditions in the limit of infinite Reynolds number (i.e., $\delta^+ \to \infty$). Solutions will be sought of the form

$$U = U_{si}(x) f_{i\infty}(y^+) \tag{20}$$

$$-\overline{uv} = R_{si}(x) r_{i\infty}(y^+) \tag{21}$$

where

$$y^+ \equiv \frac{y}{\eta} \tag{22}$$

and the length scale $\eta = \eta(x)$ remains to be determined. Note that the subscript $i\infty$ is used to distinguish the scaled velocity and Reynolds stress profiles, $f_i(y^+, \delta^+)$ and $r_i(y^+, \delta^+)$, which will be used later, from their limiting forms used here. Obviously f_i and r_i are dependent on δ^+, while $f_{i\infty}$ and $r_{i\infty}$ are not.

Substitution into equation 4 and clearing terms yields to leading order in δ^+,

$$\left[\frac{u_*^2}{U_{si}^2}\right] = \left[\frac{R_{si}}{U_{si}^2}\right] r_{i\infty} + \left[\frac{\nu}{\eta U_{si}}\right] f_{i\infty}{}' \tag{23}$$

A similarity solution exists only if η, U_{si}, and R_{si} can be determined so that all the terms in brackets have the same x-dependence; i.e.,

$$\left[\frac{u_*^2}{U_{si}^2}\right] \sim \left[\frac{R_{si}}{U_{si}^2}\right] \sim \left[\frac{\nu}{\eta U_{si}}\right] \tag{24}$$

Since there are three scaling functions to be determined, but only two independent constraints, there is some arbitrariness in their determination. A convenient choice for η is

$$\eta = \nu / U_{si} \tag{25}$$

from which it follows immediately that similarity solutions are possible only if the inner Reynolds stress scale is given by

$$R_{si} = U_{si}^2 \tag{26}$$

It is now also obvious that the inner velocity scale must be the friction velocity so that

$$U_{si} \equiv u_* \tag{27}$$

It follows that

$$\eta = \nu / u_* \tag{28}$$

$$R_{si} = u_*^2 \tag{29}$$

Thus, the integrated inner equation at infinite Reynolds number (i.e., $\delta^+ \to \infty$) reduces to

$$1 = r_{i\infty} + f_{i\infty}{}' \tag{30}$$

For finite values of δ^+. this is equation is only approximately valid because of the neglected mean convection terms.

The similarity variables derived above are the usual choices for the inner layer, and thus the *Law* [3] *of the Wall* is consistent with full similarity of the inner equations, *in the limit of infinite Reynolds number.* For any finite (but large) Reynolds number, solutions for the inner layer will retain a Reynolds number dependence (as discovered from the Pi-theorem in deriving equation 8) since the governing equations themselves do so. It is obvious then that it is equation 8 which reduces to the proper limiting form to be a similarity solution for the inner layer, and thus it must be the real Law of the Wall for finite Reynolds numbers. At finite Reynolds numbers however, it also describes the velocity profile over the entire layer. These ideas are not incompatible since in inner variables the outer layer can never be reached in the limit of infinite Reynolds number (i.e., as $\delta^+ \to \infty$, $y^+ \to \infty$ for finite values of y).

7 Full Similarity of the Outer Equations

In accordance with the Asymptotic Invariance Principle, solutions will be sought which reduce to similarity solutions of the outer momentum equation and boundary conditions in the limit of infinite Reynolds number. It is important to remember that no scaling laws will be assumed at the outset, but rather will be derived from the conditions for similarity of the equations.

For the outer equations, solutions are sought which are of the form

$$U - U_\infty = U_{so}(x) f_{o\infty}(\overline{y}) \tag{31}$$

$$-\overline{uv} = R_{so}(x) r_{o\infty}(\overline{y}) \tag{32}$$

where

$$\overline{y} = y/\delta(x) \tag{33}$$

and U_{so}, R_{so}, and δ are functions only of x. Note that extra arguments could have been included in the functional dependence of $f_{o\infty}$ and $r_{o\infty}$ to account for the effect of upstream conditions, etc. The velocity has been written as a deficit to avoid the necessity of accounting for an offset arising from viscous effects across the inner layer. This is, of course, is not a problem for the Reynolds stress since it vanishes outside the boundary layer. As in the previous section, the ∞ has been added to the subscript to distinguish $f_{o\infty}$ and $r_{o\infty}$ from the the δ^+-dependent profiles scaled with U_{so} and R_{so} used later. The V-component of velocity has been eliminated by integrating the continuity equation from the wall, thus introducing a contribution from the inner layer which vanishes identically at infinite Reynolds number.

Substitution into equation 2 and clearing terms yields

$$\left[\left(\frac{U_\infty}{U_{so}}\right)\frac{\delta}{U_{so}}\frac{dU_{so}}{dx}\right] f_{o\infty} + \left[\frac{\delta}{U_{so}}\frac{dU_{so}}{dx}\right] f_{o\infty}^2 - \left[\frac{U_\infty}{U_{so}}\frac{d\delta}{dx}\right]\overline{y} f_{o\infty}{}' - \left\{\frac{d\delta}{dx} + \left[\frac{\delta}{U_{so}}\frac{dU_{so}}{dx}\right]\right\} f_{o\infty}{}' \int_0^{\overline{y}} f_{o\infty}(\xi) d\xi = \left[\frac{R_{so}}{U_{so}^2}\right] r_{o\infty}{}'. \tag{34}$$

Note that the streamwise gradient of the normal stresses (i.e., the last term of equation 2) has been neglected for now, but with no loss of generality. (This will be shown in the next section where the normal stresses are considered individually).

[3] The word 'law' is formally incorrect since the result has been derived, and no longer depends on experimental results alone to establish its validity.

For a similarity solution to be possible, the bracketed terms must all have the same $x-$dependence (or be identically zero). Therefore, it is clear that full similarity is possible only if

$$\left(\frac{U_\infty}{U_{so}}\right)\frac{\delta}{U_{so}}\frac{dU_{so}}{dx} \sim \frac{\delta}{U_{so}}\frac{dU_{so}}{dx} \sim \left(\frac{U_\infty}{U_{so}}\right)\frac{d\delta}{dx} \sim \frac{d\delta}{dx} \sim \frac{R_{so}}{U_{so}^2}. \tag{35}$$

It follows immediately from the third and fourth conditions that

$$U_{so} = U_\infty \tag{36}$$

Since U_∞ is presumed constant, the first two conditons are identically zero and must be removed from further consideration. The remaining three can be satisfied only if (at least to within a constant of proportionality),

$$R_{so} = U_\infty^2 \frac{d\delta}{dx}, \tag{37}$$

Thus, the proper velocity scale for the velocity deficit law must be U_∞, and not u_* as suggested by Von Karman (1930) and widely utilized since (eg., Clauser 1954, Coles 1956, 1962).

Thus the limiting form of the outer equation governing the mean flow reduces to

$$-\overline{y}f_{o\infty}{}' - f_{o\infty}{}'\int_0^{\overline{y}} f_{o\infty}(\xi)d\xi = \left[\frac{R_{so}}{U_{so}^2 d\delta/dx}\right] r_{o\infty}{}'. \tag{38}$$

This equation will not receive further attention in this paper since it is not possible to close it without a turbulence model. It is important to note, however, that it has served an extremely important role since it has determined the outer scaling parameters according to the AIP, and hence the real deficit law.

The analysis above makes it clear that of the possible candidates for an outer scaling law for the velocity, only the profile represented by equation 9 is Reynolds number invariant in the limit. Therefore this must be the appropriate scaling for finite Reynolds numbers as well. (This is, of course, the whole idea behind the Asymptotic Invariance Principle.)

The old deficit profile, equation 10, can not be Reynolds number invariant in the limit (unless u_*/U_∞ is non-zero in the limit), since $F_o = (u_*/U_\infty)f_o$. In fact, since f_o is Reynolds number invariant in the limit, it is clear why F_o vanishes in this limit if $u_*/U_\infty \rightarrow 0$ (as required in the Clauser/Millikan theory and as derived below). This is precisely objection (i) registered in the Introduction.

The Reynolds stress scale, on the other hand, is *not* U_∞^2, but an entirely different scale depending on the growth rate of the boundary layer, $d\delta/dx$. In effect, $d\delta/dx$ is acting as a Reynolds number dependent correlation coefficient, just as for free shear flows (George 1989). This will be shown later to be related to the fact that as the Reynolds number increases, less and less of the energy is dissipated at the scales at which the Reynolds stress is adding energy to the flow so they become effectively inviscid (v. George 1995). It will also be shown below that R_{so} can be determined by matching the outer Reynolds stress to the inner Reynolds stress. The need for such a matching is intuitively obvious, since the only non-zero boundary condition on the Reynolds stress in the outer flow is that imposed by the inner.

Millikan (1938) and others have objected to the type of similarity analysis employed here as leading to unphysical results for the boundary layer. Certainly there is nothing unphysical about the velocity deficit law using U_∞ in and of itself, and a case for such a deficit law could have been made, even with the data available at the time (as was suggested earlier). Thus the fundamental basis for this objection must have been the condition on the Reynolds stress. However, this would have been a problem only if it were also required or *assumed at the outset* that $R_{so} = U_{so}^2$, for then it would have also been necessary that $d\delta/dx = constant$. Since the boundary layer was believed not to grow linearly, Millikan (and many before and after him as well) was forced to conclude that full self-preservation (in the assumed sense) was not possible, and therefore had to settle for a *locally* self-preserving solution.

George (1989), however, pointed out that (contrary to the conventional wisdom of self-preservation presented in texts) there is no reason *a priori* to insist that $R_{so} = U_{so}^2$. If this arbitrary requirement is relaxed, then there is no longer the requirement for linear growth, and both equation 37 and similarity become tenable. In fact, these conditions require that the outer flow be governed by two velocity scales, U_∞ and a second governing the Reynolds stress which is determined by the boundary conditions imposed on the Reynolds stress by the inner layer. It will be shown below that the inner and outer Reynolds stresses can overlap asymptotically only if

$$R_{so} \sim U_\infty^2 \frac{d\delta}{dx} \sim u_*^2 \tag{39}$$

which resembles closely the momentum integral equation, both a surprising and gratifying result. More will be said on this relationship later.

That the outer (and inner) equations admit to similarity solutions (in the sense of George 1989) should come as no surprise to the experimentalists who have long recognized their ability to collapse the outer mean velocity data with only U_∞ and δ. Hinze 1976 and Schlichting 1968, for example, show profiles normalized by U/U_∞ and plotted as a function of y/δ. Even the fact that the outer Reynolds stress scales with u_* (but only to first order) is in accord with common practice, since it is assumed in the old theory — but in a way which could not account for the observed weak dependence on Reynolds number. Thus one can speculate that Millikan's conclusions might have been quite different had he (and several generations after him) not been locked-in to the too restrictive idea of self-preservation (i.e., single length and velocity scales).

8 Scaling of the Other Turbulence Quantities

For the inner layer, there is only one velocity scale, u_*, which enters the *single point* equations; therefore all *single point* statistical quantities must scale with it. This is, of course, the conventional wisdom, but with an important difference: *The inner layer does not include the overlap layer* — the region between the inner and outer regions — which is Reynolds number dependent. This is contrary to the conventional wisdom of including the overlap layer as part of the wall layer. But since the inner and outer scales are different, the dependent variables in the overlap layer must be expected to be functions of both, *and thus Reynolds number dependent.* (Note that different considerations must be applied to the multi-point equations since conditions at a point can depend on those at another, and in particular those at a distance.)

From the preceding analysis, it is apparent that the outer layer at finite Reynolds numbers is governed by not one, but two velocity scales. In particular, the mean velocity and its gradients scale with U_∞, while the Reynolds shear stress scales with $U_\infty^2 d\delta/dx \sim u_*^2$. Therefore it is not immediately obvious how the remaining turbulence quantities should scale. In particular, do they scale with U_∞ or u_*, or both? If the latter, then quantities scaled in the traditional way with only one of them will exhibit a Reynolds number dependence and will not collapse. (Note that if the ratio of velocity scales, u_*/U_∞, is asymptotically constant, this Reynolds number dependence would appear to reduce with increasing distance downstream and could lead to the erroneous conclusion that certain quantities scaled with only one of them take longer to reach equilibrium than others.)

In view of the possible similarity of the outer equations for the mean flow, it is reasonable to inquire whether the equations for other turbulence quantities also admit to fully similar solutions. For the outer part of the boundary layer at high Reynolds number, the equation for $< u^2 >$ can be written (Tennekes and Lumley 1972) as

$$\begin{aligned} U\frac{\partial < u^2 >}{\partial x} + V\frac{\partial < u^2 >}{\partial y} &= 2 < p\frac{\partial u}{\partial x} > \\ &+ \frac{\partial}{\partial y}\{- < u^2 v >\} - 2 < uv > \frac{\partial U}{\partial y} - 2\epsilon_u \end{aligned} \tag{40}$$

where ϵ_u is the energy dissipation rate for $< u^2 >$ and the viscous transport term has been neglected. It might be noted that an order of magnitude analysis reveals the mean convection and turbulence transport terms to be of second order in the turbulence intensity u'/U, so to first order the equation reduces to simply a balance among the production, dissipation and pressure strain rate terms. It could be argued that these second order terms should be neglected in the subsequent analysis, cf. Townsend (1975). It is precisely these second order terms, however, that distinguish one boundary layer type flow from another, or from homogeneous flows (like channels and pipes) for that matter. Therefore, for a theory which purports to represent growing shear layers like the wall jet, they must be retained.

By considering similarity forms for the new moments like

$$\frac{1}{2} < u^2 > = K_u(x)k(\overline{y}) \tag{41}$$

$$< p\frac{\partial u}{\partial x} > = P_u(x)p_u(\overline{y}) \tag{42}$$

$$-\frac{1}{2} < u^2 v > = T_{u^2v}(x)t(\overline{y}) \tag{43}$$

$$\epsilon_u = D_u(x)d(\overline{y}) \tag{44}$$

and using $R_s = U_\infty^2 d\delta/dx$, it is easy to show that similarity of the $< u^2 >$-equation is possible only if

$$K_u \sim U_\infty^2 \tag{45}$$

$$P_u \sim \frac{U_\infty^3}{\delta}\frac{d\delta}{dx} \sim \frac{U_\infty u_*^2}{\delta} \tag{46}$$

$$T_{u^2v} \sim U_\infty^3 \frac{d\delta}{dx} \sim U_\infty u_*^2 \tag{47}$$

$$D_u \sim \frac{U_\infty^3}{\delta}\frac{d\delta}{dx} \sim \frac{U_\infty u_*^2}{\delta} \tag{48}$$

All of these are somewhat surprising: The first (even though a second moment like the Reynolds stress) because the factor of $d\delta/dx$ is absent; the second, third and fourth because it is present. The mixed forms using u_* and U_∞ instead of $d\delta/dx$ should be especially useful for scaling experimental data at low to moderate Reynolds numbers where u_*/U_∞ shows considerable variation.

Similar equations can be written for the $< v^2 >$ and $< w^2 >$; i.e.,

$$\begin{aligned} U\frac{\partial < v^2 >}{\partial x} + V\frac{\partial < v^2 >}{\partial y} &= 2 < p\frac{\partial v}{\partial y} > \\ &+ \frac{\partial}{\partial y}\left\{- < v^3 > -2 < pv >\right\} - 2\epsilon_v \end{aligned} \tag{49}$$

and

$$\begin{aligned} U\frac{\partial < w^2 >}{\partial x} + V\frac{\partial < w^2 >}{\partial y} &= 2 < p\frac{\partial w}{\partial z} > \\ &+ \frac{\partial}{\partial y}\left\{- < w^2 v >\right\} - 2\epsilon_w \end{aligned} \tag{50}$$

When each of the terms in these equations is expressed in similarity variables, the resulting similarity conditions are:

$$D_v \sim P_v \sim \frac{U_\infty K_v}{\delta}\frac{d\delta}{dx} \tag{51}$$

$$D_w \sim P_w \sim \frac{U_\infty K_w}{\delta}\frac{d\delta}{dx} \tag{52}$$

$$T_{v^3} \sim \frac{U_\infty K_v}{\delta}\frac{d\delta}{dx} \tag{53}$$

$$T_{w^2 v} \sim \frac{U_\infty K_w}{\delta}\frac{d\delta}{dx} \tag{54}$$

There is an additional equation which must be accounted for; namely that the sum of the pressure strain-rate terms in the component energy equations be zero (from continuity). Thus, in similarity variables,

$$P_u(x)p_u(\overline{y}) + P_v(x)p_v(\overline{y}) + P_w(x)p_w(\overline{y}) = 0 \tag{55}$$

This can be true for all $\overline{y}$ only if

$$P_u \sim P_v \sim P_w \tag{56}$$

An immediate consequence from equations 51 and 52 is that

$$D_u \sim D_v \sim D_w \sim D_s \sim \frac{U_\infty^3}{\delta}\frac{d\delta}{dx} \sim \frac{U_\infty u_*^2}{\delta} \tag{57}$$

where D_s is the scale for the entire dissipation, and

$$K_u \sim K_v \sim K_w \sim U_\infty^2 \tag{58}$$

Thus all of the Reynolds *normal* stresses scale with U_∞^2, and not with u_*^2 like the Reynolds *shear* stress. Note that this does *not* imply that the normal stresses are the same order of magnitude as U_∞^2 which clearly can not be the case, only that there functional dependence on x is the same. It is easy to show that relations of equation 58 are consistent with similarity of the neglected normal stress terms in the mean momentum equation 34; hence the equation for the mean flow is consistent with similarity to the second order, exactly as is being required here.

The remaining equation for the Reynolds shear stress is given by

$$\begin{aligned} U\frac{\partial <uv>}{\partial x} + V\frac{\partial <uv>}{\partial y} &= <p\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)> \\ &+ \frac{\partial}{\partial y}\{-<uv^2>\} - <v^2> \frac{\partial U}{\partial y} \end{aligned} \tag{59}$$

This does not introduce any new similarity functions, but it does impose a surprising constraint on the ones which exist already, namely,

$$K_v \sim R_{so}\frac{d\delta}{dx} \sim U_\infty^2 \left(\frac{d\delta}{dx}\right)^2 \sim u_*^2 \frac{d\delta}{dx} \tag{60}$$

There is an apparent contradiction between equation 60 and equation 58. However, R_{so} is only *asymptotically* equal to u_*^2 (see Section 13 below), so the outer Reynolds stress scale only evolves to this value with increasing Reynolds number. Obviously, the two conditions together require that in the limit of infinite Reynolds number,

$$\frac{d\delta}{dx} \sim \frac{u_*^2}{U_\infty^2} \sim constant \tag{61}$$

It will be argued later that the constant is, in fact, zero, so obviously no scaling can work at finite Reynolds number, but can only be approached asymptotically. The full implications of this will be considered in more detail in section 17.

From equations 57 and 48 it follows that

$$D_s \sim \frac{U_\infty^3}{\delta} \tag{62}$$

The relations given by equations 58 and 62 were assumed without proof in the George 1989 analysis of free shear flows. The additional constraint imposed by equation 56 was not derived, however, and arises from the additional information provided by the pressure strain-rate terms.

Before leaving this section it should be noted that conditions for similarity of the turbulence moments also give a clue as to when, if ever, this asymptotic state might be achieved. Similarity solutions of the Reynolds stress equations are possible only when $D_s(x) \sim U_s^3/\delta$ where $U_s = U_\infty$ for the boundary layer. There are only two possibilities for this to occur (George 1995):

- i) Either the local Reynolds number of the flow is constant so that the effect of viscosity on the energy containing eddies (and those producing the Reynolds stress as well) does not vary with downstream distance; or

- ii) The local turbulence Reynolds number is high enough so that the Reynolds stresses are effectively inviscid and the relation $\epsilon \sim q^3/L$ is approximately valid (for a *physical* length $L \sim \delta$).

Unlike some flows (like the axisymmetric jet or plane wake) where the local Reynolds number is constant, for the boundary layer it continues to increase with downstream distance. Therefore the only possibility for similarity at the level of the Reynolds stresses is (ii). Thus similarity in the outer boundary layer at the level of the second order moments can occur only when the *turbulence* Reynolds number is large enough.

Since the local Reynolds number for the boundary layer continues to increase with increasing downstream distance, the similarity state will eventually be reached. The higher the unit Reynolds number of the flow (U_∞/ν), the smaller the value of x at which similarity of the second order moments will be realized. (Section 11 below will attempt to establish approximate bounds where each part of the boundary layer can be Reynolds number independent.) It is worth noting that there appears to be nothing in the equations to indicate whether the similarity state achieved in the outer part of the boundary layer is independent of upstream effects.

9 The Overlap Layer: An Application of Near-Asymptotics

It is obvious that since both the outer and inner profiles are non-dimensional profiles with different scales and the ratio of the scales is Reynolds number dependent, then any region between the two similarity regimes cannot be Reynolds number independent, except possibly in the limit. The actual mean velocity profile at any Reynolds number, however, is the average of the instantaneous solutions to the Navier-Stokes equations and boundary conditions. And this profile, whether determined from a real flow, DNS simulation, or not at all, exists, at least in principle, and is valid everywhere regardless of how it is scaled. Therefore both scaled forms of this solution, $f_i(y^+, \delta^+)$ and $f_o(\overline{y}, \delta^+)$ (equations 8 and 9 respectively), represent the velocity everywhere, at least as long as the Reynolds number is finite. In fact, the parameter δ^+ uniquely labels the fanning out of the inner scaled profiles in the outer region and the outer scaled profiles near the wall (e.g., Figures 1, 2 and 3 of Part II).

Thus, f_i and f_o are quite unlike their limiting forms, $f_{i\infty}$ and $f_{o\infty}$, which are only infinite Reynolds number solutions for the inner and outer equations respectively. If f_i and f_o are considered instead of $f_{i\infty}$ and $f_{o\infty}$ (as is usually done), the problem of determining whether an overlap region exists is quite different from the usual asymptotic matching where infinite Reynolds number inner and outer solutions are extended and matched in an overlap region if one exists. Therefore, the objective here is *not* to see if f_i and f_o overlap and match them if they do. Rather, it is to see *whether the fact that these scaled finite Reynolds number solutions (to the whole flow) degenerate at infinite Reynolds number in different ways can be used to determine their functional forms in the common region they describe in the limit.* The methodology outlined below (termed *Near-Asymptotics*) is believed to be new, but is necessary because the traditional approach cannot account for the possibility of the matching parameter tending to zero, as might be the case.

The fact that analytical forms for these Reynolds number dependent solutions are not available and are only known *in principal* turns out not to be a significant handicap. There are several pieces of information about the two profiles which can be utilized without further assumptions. They are:

- First, since both inner and outer forms of the velocity profile must describe the flow everywhere as long as the ratio of length scales, $\delta^+ = \delta/\eta = \delta^+$, is finite, it follows from equations 8 and 9 that

$$1 + f_o(\overline{y}, \delta^+) = g(\delta^+) f_i(y^+, \delta^+) \tag{63}$$

  where $g(\delta^+) = u_*/U_\infty$ is defined by equation 5.

- Second, for finite values of δ^+, the velocity derivatives from both inner and outer forms of the velocity must also be the same everywhere. It is easy to show that this requires that

$$\frac{\overline{y}}{1+f_o}\frac{df_o}{d\overline{y}} = \frac{y^+}{f_i}\frac{df_i}{dy^+} \tag{64}$$

for all values of δ^+ and y.

- Third, in the limit, both f_o and f_i must become asymptotically independent of δ^+. Thus $f_o(\overline{y}, \delta^+) \rightarrow f_{o\infty}(\overline{y})$ only, and $f_i(y^+, \delta^+) \rightarrow f_{i\infty}(y^+)$ only as $\delta^+ \rightarrow \infty$ or otherwise the velocity scales have been incorrectly chosen. (This is, in fact, the *Asymptotic Invariance Principle.*)

Now the problem is that *in the limit* as $\delta^+ \rightarrow \infty$, the outer form fails to account for the behavior close to the wall while the inner fails to describe the behavior away from it. The question then is: In this limit (as well as for all finite values approaching it) does there exist an "overlap" region where equation 63 is still valid? Since both δ and η are increasing with streamwise distance along the surface, this "overlap" region will not only increase in extent when measured in either inner or outer coordinates, it will move farther from the wall in actual physical variables. (Note that this is quite different from pipe and channel flows in which the overlap layer remains at fixed distance from the wall for all x because of the streamwise homogeneity, as long as the external parameters are fixed.)

The question of whether there is a common region of validity can be investigated by examining how rapidly f_o and f_i are changing with δ^+. From the Taylor expansion about a fixed value of δ^+,

$$\frac{f_i(y^+;\delta^+ + \Delta\delta^+) - f_i(y^+;\delta^+)}{\Delta\delta^+ f_i(y^+;\delta^+)} \approx \frac{1}{f_i(y^+;\delta^+)}\frac{\partial f_i(y^+;\delta^+)}{\partial\delta^+}\Big|_{y^+} \equiv S_i(\delta^+, y^+) \tag{65}$$

and

$$\frac{f_o(\overline{y};\delta^+ + \Delta\delta^+) - f_o(\overline{y};\delta^+)}{\Delta\delta^+ f_o(\overline{y};\delta^+)} \approx \frac{1}{f_o(\overline{y};\delta^+)}\frac{\partial f_o(\overline{y};\delta^+)}{\partial\delta^+}\Big|_{\overline{y}} \equiv S_o(\delta^+, \overline{y}) \tag{66}$$

Thus S_i and S_o are measures of the Reynolds number dependencies of f_i and f_o respectively. Both vanish identically in the limit as $\delta^+ \rightarrow \infty$. If ${y^+}_{max}$ denotes a location where outer flow effects begin to be strongly felt on the inner scaled profile, then for $y^+ < {y^+}_{max}$, S_i should be much less than unity (or else the inner scaling is not very useful). Similarly, if $\overline{y}_{min}$ measures the location where viscous effects begin to be strongly felt (e.g., as the linear velocity region near the wall is approached), then S_o should be small for $\overline{y} > \overline{y}_{min}$. Obviously either S_i or S_o should increase as these limits are approached. Outside these limits, one or the other should increase dramatically.

The quantities S_i and S_o can, in fact, be used to provide a formal definition of an "overlap" region where both scaling laws are valid. Since S_i will increase drastically for large values of y for given δ^+ and S_o will increase for small values of y, an "overlap" region exists only if there exists a region for which both S_i and S_o remain small simultaneously. In the following paragraphs, this condition will be used in conjunction with equation 63 to derive the functional form of the velocity in the overlap region *at finite Reynolds number*, hence the term 'Near-Asymptotics'. Obviously there is a very close relation between the idea of Near-Asymptotics and Intermediate Asymptotics (Barenblatt 1978), the difference being that the former is carried out at finite Reynolds number.

Because of the movement of the matched layer away from the wall with increasing x, it is convenient and necessary to introduce an intermediate variable $\tilde{y}$ which can be fixed in the overlap region all the way to the limit, regardless of what is happening in physical space (v. Cole and Kevorkian 1981). A definition of $\tilde{y}$ which accomplishes this is given by

$$\tilde{y} = y^+ {\delta^+}^{-n} \tag{67}$$

or

$$y^+ = \tilde{y} {\delta^+}^{n} \tag{68}$$

Since $\overline{y} = y^+/\delta^+$, it follows that

$$\overline{y} = \tilde{y} {\delta^+}^{n-1} \tag{69}$$

For all values of n satisfying $0 < n < 1$, $\tilde{y}$ can remain fixed in the limit as $\delta^+ \to \infty$ while $\overline{y} \to 0$ and $y^+ \to \infty$. Substituting these into equation 63 yields the matching condition on the velocity in terms of the intermediate variable as

$$1 + f_o(\tilde{y} {\delta^+}^{n-1}, \delta^+) = g(\delta^+) f_i(\tilde{y} {\delta^+}^{n}, \delta^+) \tag{70}$$

Now equation 70 can be differentiated with respect to δ^+ *for fixed* $\tilde{y}$ to yield equations which explicitly include S_i and S_o. The result is

$$\left.\frac{\partial(1+f_o)}{\partial \overline{y}}\right|_{\delta^+} \left.\frac{\partial \overline{y}}{\partial \delta^+}\right|_{\tilde{y}} + \left.\frac{\partial(1+f_o)}{\partial \delta^+}\right|_{\overline{y}} = \frac{dg}{d\delta^+} f_i + g \left\{ \left.\frac{\partial f_i}{\partial y^+}\right|_{\delta^+} \left.\frac{\partial y^+}{\partial \delta^+}\right|_{\tilde{y}} + \left.\frac{\partial f_i}{\partial \delta^+}\right|_{y^+} \right\} \tag{71}$$

Carrying out the indicated differentiation of y^+ and $\overline{y}$ by δ^+ (for fixed $\tilde{y}$), and multiplying by $\delta^+/(1+f_o)$ yields (after some rearranging)

$$(n-1)\frac{\overline{y}}{(1+f_o)} \left.\frac{\partial(1+f_o)}{\partial \overline{y}}\right|_{\delta^+} - n \frac{y^+}{f_i} \left.\frac{\partial f_i}{\partial y^+}\right|_{\delta^+} = \frac{\delta^+}{g}\frac{dg}{d\delta^+} + \delta^+ \left\{ \frac{1}{f_i} \left.\frac{\partial f_i}{\partial \delta^+}\right|_{y^+} - \frac{1}{1+f_o} \left.\frac{\partial(1+f_o)}{\partial \delta^+}\right|_{\overline{y}} \right\} \tag{72}$$

It follows immediately from equation 64 that

$$\frac{\overline{y}}{1+f_o} \left.\frac{\partial(1+f_o)}{\partial \overline{y}}\right|_{\delta^+} = -\frac{\delta^+}{g}\frac{dg}{d\delta^+} - \delta^+ \left\{ \frac{1}{f_i} \left.\frac{\partial f_i}{\partial \delta^+}\right|_{y^+} - \frac{1}{1+f_o} \left.\frac{\partial(1+f_o)}{\partial \delta^+}\right|_{\overline{y}} \right\} \tag{73}$$

Equation 73 can be rewritten as

$$\frac{\overline{y}}{1+f_o} \left.\frac{\partial(1+f_o)}{\partial \overline{y}}\right|_{\delta^+} = \gamma(\delta^+) - \delta^+(S_i - S_o) \tag{74}$$

where $\gamma = \gamma(\delta^+)$ is defined by

$$\gamma(\delta^+) \equiv -\frac{\delta^+}{g}\frac{dg}{d\delta^+} = \frac{d\ln(1/g)}{d\ln \delta^+} \tag{75}$$

Note that the first term on the right hand side of equation 74 is at most a function of δ^+ alone, while the second term contains all of the residual y-dependence.

Now it is clear that if both

$$\delta^+ |S_o| << \gamma \tag{76}$$

and

$$\delta^+ |S_i| << \gamma \tag{77}$$

then the first term on the right-hand side of equation 74 dominates. If $\gamma \to 0$, then the inequalities are still satisfied as long as the left hand side does so more rapidly than γ. Note that a much weaker condition can be applied which yields the same result; namely that both inner and outer scaled profiles have the same dependence on δ^+, i.e., $S_i = S_o$ in the overlap range so γ is the only term remaining. (The authors are grateful to Professor R. Karlsson of KTH/Stockholm for pointing this out.) If these inequalities are satisfied over some range in y, then to leading order, equation 73 can be written as

$$\frac{\overline{y}}{1+f_o^{(1)}} \frac{\partial(1+f_o^{(1)})}{\partial \overline{y}}\bigg|_{\delta+} = \gamma(\delta^+) \tag{78}$$

The solution to equation 78 can be denoted as $f_o^{(1)}$ since it represents a first order approximation to f_o. It is *not*, however, simply the same as $f_{o\infty}$ because of the δ^+ dependence of γ, but reduces to it in the limit. Thus, by regrouping into the leading term all of the y-independent contributions, the method applied here has yielded a more general result than the customary expansion about infinite Reynolds number. (It is also easy to see why the usual matching of infinite Reynolds number inner and outer solutions will not work since the limiting value of γ might be zero.)

From equations 64 and 78, it follows that

$$\frac{y^+}{f_i^{(1)}} \frac{\partial f_i^{(1)}}{\partial y^+}\bigg|_{\delta+} = \gamma(\delta^+) \tag{79}$$

An interesting feature of these first order solutions is that the inequalities given by equations 76 and 77 determine the limits of validity of both equations 78 and 79 since either S_o or S_i will be large outside the overlap region. Clearly the extent of this region will increase as the Reynolds number (or δ^+) increases.

Both equations 79 and 78 must be invariant to transformations of the form $y \to y + a$ where a is arbitrary since the equation must be valid for any choice of the origin of y (The authors are grateful to Dr. Martin Oberlack, of ITM/Aachen, for this insight). Therefore, the most general solutions are of the form:

$$1 + f_o^{(1)}(\overline{y}, \delta^+) = C_o(\delta^+)\overline{y} + \overline{a}^{\gamma(\delta^+)} \tag{80}$$

$$f_i^{(1)}(y^+, \delta^+) = C_i(\delta^+)y^+ + a^{+\gamma(\delta^+)} \tag{81}$$

or in physical variables,

$$\frac{U - U_\infty}{U_\infty} = C_o(\frac{y+a}{\delta})^\gamma \tag{82}$$

$$\frac{U}{u_*} = C_i(\frac{y+a}{\eta})^\gamma \tag{83}$$

Thus the velocity profile in the overlap layer is a power law with coefficients and exponent which depend only on Reynolds number, δ^+. The parameter a^+ will be found to be nearly constant and related to the mesolayer discussed below. [4] In the remainder of this paper, the superscript '(1)' will be dropped; however it is these first order solutions that are being referred to unless otherwise stated.

[4] Earlier versions of this theory (George et al. 1992, George and Castillo 1993) included additive constants which were believed to be zero only on experimental grounds. The derivation here makes it clear that these constants are indeed zero.

The relation between u_* and U_∞ follows immediately from equation 63; i.e.,

$$\sqrt{\frac{c_f}{2}} = \frac{u_*}{U_\infty} = g(\delta^+) = \frac{C_o(\delta^+)}{C_i(\delta^+)} \delta^{+^{-\gamma(\delta^+)}} \tag{84}$$

Thus the friction law is also a power law entirely determined by the velocity parameters for the overlap region. However, equation 75 must also be satisfied. Substituting equation 84 into equation 75 implies that γ, C_o, and C_i are constrained by

$$ln\delta^+ \frac{d\gamma}{d\ln\delta^+} = \frac{d\ln[C_o/C_i]}{d\ln\delta^+} \tag{85}$$

It is immediately obvious that this constraint equation must be invariant to scale transformations of the form $\delta^+ \to D\delta^+$ since the physical choice of δ^+ must be arbitrary (e.g., δ_{99}, δ_{95}, etc.). Thus the Reynolds number dependence of γ and C_o/C_i must also independent of the particular choice of δ, since any other choice (say from δ_{99}) would simply be reflected in the constant coefficient D. This fact will be of considerable importance in relating the boundary layer parameters to other way bounded flows as well (v. George et al. 1997). Also obvious from equation 85 is that both γ and C_o/C_i can be most conveniently expressed as functions of $\ln\delta^+$.

Equation 85 is exactly the criterion for the neglected terms in equation 73 to vanish identically (i.e., $S_i - S_o \equiv 0$). Therefore the solution represented by equations 80 – 85 is, indeed, the first order solution for the velocity profile in the overlap layer at *finite*, but large, Reynolds number. Clearly when y^+ is too big or $\overline{y}$ is too small for a given value of δ^+, the inequalities of equations 76 and 77 cannot be satisfied. Since all the derivatives with respect to δ^+ must vanish as $\delta^+ \to \infty$ (the A.I.P.), the outer range of the inner overlap solution is unbounded in the limit, while the inner range of the outer is bounded only by zero.

The parameters $C_i(\delta^+)$, $C_o(\delta^+)$, $\gamma(\delta^+)$ and a as well as the constant D must be determined either empirically or from a closure model for the turbulence. However they are determined, the results must be consistent with equation 85. Also, equations 80 and 81 must be asymptotically independent of Reynolds number, since f_i and f_o are. Therefore the coefficients and exponent must be asymptotically constant; i.e.,

$$\begin{aligned} \gamma(\delta^+) &\to \gamma_\infty \\ C_o(\delta^+) &\to C_{o\infty} \\ C_i(\delta^+) &\to C_{i\infty} \end{aligned}$$

as $\delta^+ \to \infty$. These conditions are powerful physical constraints and together with equation 85 will be seen to rule out some functional forms for γ, like that suggested by Barenblatt 1993 for example (see below). Therefore it is important to note that they are a direct consequence of the AIP and the assumption that scaling laws should correspond to similarity solutions of the equations of motion.

It is convenient to write the solution to equation 85 as

$$\frac{C_o}{C_i} = \exp[(\gamma - \gamma_\infty)\ln\delta^+ + h] \tag{86}$$

where $h = h(\delta^+)$ remains to be determined, but must satisfy

$$\gamma - \gamma_\infty = -\delta^+ \frac{dh}{d\delta^+} = -\frac{dh}{d\ln\delta^+} \tag{87}$$

The advantage of this form of the solution is easily seen by substituting equation 86 into equation 95 to obtain

$$\frac{u_*}{U_\infty} = \exp[-\gamma_\infty \ln \delta^+ + h] \tag{88}$$

Thus u_*/U_∞ is entirely determined by γ_∞ and $h(\delta^+)$.

It is easy to show the conditions that both $C_{o\infty}$ and $C_{i\infty}$ be finite and non-zero require that:
Either

- C_o, C_i and γ remain constant always;

or

- (i) $\gamma \rightarrow \gamma_\infty$ faster than $1/\ln \delta^+ \rightarrow 0$

and

- (ii) $h(\delta^+) \rightarrow h_\infty = constant$.

It follows immediately that

$$\frac{C_{o\infty}}{C_{i\infty}} = exp[h_\infty] \tag{89}$$

Note that condition (i) together with equation 87 requires that $dh/d\ln\delta^+ \rightarrow 0$ faster than $1/\ln\delta^+ \rightarrow 0$.

Condition (ii) rules out solutions of the form suggested by Barenblatt 1993 who proposed power law profiles with $\gamma = a/\ln\delta^+$ for which $h = \ln b - a \ln\ln\delta^+$ where $\ln b$ is the integration constant. Obviously this h is unbounded in the limit as $\delta^+ \rightarrow \infty$. Substitution into equation 86 yields $C_o/C_i = b(e/\ln\delta^+)^a$. Thus, either $C_o \rightarrow 0$ or $C_i \rightarrow \infty$ or both. Both of these are unacceptable alternatives in that they are inconsistent with similarity of even the mean velocity. [5]

It is interesting to examine the relation between the asymptotic value of γ and u_*/U_∞. Since γ must be asymptotically independent of δ^+, the only possible values for γ_∞ are either a finite constant, or zero. For the former, $u_*/U_\infty \rightarrow 0$, while for the latter the limiting value is finite and non-zero. Note that both of these satisfy the condition from Section 8 for similarity of the Reynolds stress equations in the outer layer (i.e., $d\delta/dx \sim constant$).

A zero limit for γ itself can be considered by using Equation 88 to obtain

$$\frac{u_*}{U_\infty} = \exp[h] \tag{90}$$

Recall that if $\gamma \rightarrow 0$ faster than $1/\ln\delta^+$ and $h \rightarrow const$ as required, this insures a finite asymptotic value of u_*/U_∞. Hence there is no question about whether the turbulence moments in the *outer* layer approach a state of asymptotic similarity; they do, since the limiting value of both $d\theta/dx$ and $d\delta/dx$ is finite. A finite and non-zero limiting value for u_*/U_∞ is certainly contrary to traditional thinking, and would have important implications for the engineer.

So there would seem to be a strong argument for $\gamma \rightarrow 0$. But this presents another problem. In the overlap region in the limit of infinite Reynolds number, the production of turbulence energy is exactly balanced by

[5] Barenblatt's form does produce a logarithmic drag law which is desirable for channel flow, but not necessarily for a boundary layer. A logarithmic drag law can be obtained for channel flow in another way as shown in Appendix I.

the rate of dissipation. Thus, in inner variables, $\epsilon^+ = P^+ = \gamma C_i {y^+}^{\gamma-1}$ since $< -uv >= u_*^2$ in this limit. If there is indeed an energy dissipation law (Frisch 1995) which demands that the *local* rate of dissipation be finite and non-zero in the limit of infinite Reynolds number, then γ_∞ must also be finite and non-zero since C_i must be finite and non-zero for similarity as noted earlier. In fact, the data shown in Part II are most consistent with a non-zero value of γ_∞, but the experimental evidence is not conclusive.

In Part II it will be found on empirical grounds that the variation of $\gamma - \gamma_\infty$ and C_o/C_i with δ^+ is described to an very good approximation by

$$h - h_\infty = \frac{A}{(\ln \delta^+)^\alpha} \tag{91}$$

where $\alpha = 0.46$, $A = 2.90$ and $D = 1$ for δ chosen to be δ_{99}. This can easily be shown to satisfy the constraints above. It follows immediately from equations 87 and 86 that

$$\gamma - \gamma_\infty = \frac{\alpha A}{(\ln \delta^+)^{1+\alpha}} \tag{92}$$

$$\frac{C_o}{C_i} = \frac{C_{o\infty}}{C_{i\infty}} \exp[(1+\alpha)A/(\ln \delta^+)^\alpha] \tag{93}$$

and

$$\frac{u_*}{U_m} = \frac{C_{o\infty}}{C_{i\infty}} [\delta^+]^{-\gamma_\infty} \exp[A/(\ln \delta^+)^\alpha] \tag{94}$$

It will be shown in Part II that the data are consistent with $C_{o\infty} = 0.897$, $C_{i\infty} = 55$ and $\gamma_\infty = 0.0362$.

Before leaving this section it is reassuring to note that in flows where the inner and outer velocity scales are the same (as for the channel flow of Appendix I), the procedure utilized above leads to the familiar logarithmic profiles with coefficients which are Reynolds number dependent, but asymptotically constant. The possibility of Reynolds number dependent parameters for the log law in pipe and channel flows is certainly contrary to the prevailing wisdom, but consistent with widespread speculation in the experimental community over many decades. Finally, both the power law profile proposed here and the log profile have recently been shown by Oberlack (1996) to be consistent with a Lie group analysis of the governing equations for parallel shear flow. The AIP and Near-Asymptotic analysis used here has supplied the information missing from the Lie group analysis; namely, *where* at least two of the particular analytical solutions apply.

10 A New Friction Law

The relation between u_*/U_∞ and δ^+ has already been established by equation 84. This can be rewritten as

$$\frac{u_*}{U_\infty} = \frac{C_o}{C_i} {\delta^+}^{-\gamma} = \frac{C_o}{C_i} e^{-\gamma \ln \delta^+} \tag{95}$$

The asymptotic behavior of u_*/U_∞ obviously is determined by both γ and the ratio C_o/C_i, which are themselves interrelated by equation 85.

It is easy to see from the above conditions on γ and $h(\delta^+)$ that the asymptotic friction law is a power law given by

$$\frac{u_*}{U_\infty} \to \frac{C_{o\infty}}{C_{i\infty}} {\delta^+}^{-\gamma_\infty} \tag{96}$$

Some idea of when this limit might be achieved can be obtained by expanding the exponential of equation 94 in powers of $A/(\ln(\delta^+)^\alpha$ to obtain

$$exp[A/(\ln\delta^+)^\alpha] = 1 + \frac{A}{(\ln\delta^+)^\alpha} + \ldots \tag{97}$$

Clearly the second term must be negligible for the power law behavior to dominate; thus the limiting power law behavior is obtained when

$$\ln\delta^+ >> [A]^{1/\alpha} \tag{98}$$

For the values A and α cited above this would require $\delta^+ >> 2.4 \times 10^4$, or at least an order of magnitude above the existing experiments ($\delta^+ \approx 16,000$ is the highest).

The friction laws written above all use u_* on both sides of the equation. This can be cast in an alternative form by eliminating the dependence of the right-hand side of equation 84 on u_*; i.e.,

$$\frac{u_*}{U_\infty} = \left(\frac{C_o}{C_i}\right)^{1/1+\gamma} \left(\frac{U_\infty\delta}{\nu}\right)^{-\gamma/(1+\gamma)} \tag{99}$$

or

$$c_f = 2\left(\frac{C_o}{C_i}\right)^{2/(1+\gamma)} \left(\frac{U_\infty\delta}{\nu}\right)^{-2\gamma/(1+\gamma)} \tag{100}$$

Unfortunately, because C_o/C_i is itself a function of Reynolds number, this form is less useful than it might appear to be. In sections 15 and 24 the relation of R_δ to R_θ and R_{δ_*} will be determined so that the friction law can be expressed in terms of any of the convenient Reynolds numbers.

11 The Effect of Reynolds Number

The overlap layer identified in the preceding sections can be related directly to the averaged equations for the mean flow and the Reynolds stresses. Of particular interest is the question of how large the Reynolds number must be before the boundary layer begins to show the characteristics of the asymptotic state.

The averaged momentum equation from about $y^+ > 30$ out to $\overline{y} < 0.1$ is given approximately by

$$0 = -\frac{\partial < uv >}{\partial y} \tag{101}$$

It has no explicit Reynolds number dependence; and the Reynolds shear stress is effectively constant throughout this region. Unfortunately many low Reynolds number experiments do not have a region where this is even approximately true because the convection terms are not truly negligible. Hence it is unreasonable to expect these experimental profiles to display any of the characteristics of the overlap described above, except possibly in combination with the characteristics of the other regions. (For example, the composite velocity profile of section 14 can be used to obtain the Reynolds stress by integrating the complete momentum equation from the wall.)

Even when there is a region of reasonably constant Reynolds stress, however, this is not the entire story because of the Reynolds number dependence of $< -uv >$ itself. Recall that the parameters D_i, D_o, and β (and the velocity parameters C_o, C_i and γ as well) were only asymptotically constant, and only in the limit did $\beta \to 0$. The origin of this weak Reynolds number dependence can be seen by considering the Reynolds transport equations. For this "constant shear stress region", the viscous diffusion and mean convection terms are negligible (as in the mean momentum equation), so the equations reduce approximately to (Tennekes and Lumley 1972),

$$0 \approx -(< p\frac{\partial u_i}{\partial x_k} > + < p\frac{\partial u_k}{\partial x_i} >) - \left[< u_i u_2 > \frac{\partial U_k}{\partial x_2} + < u_k u_2 > \frac{\partial U_i}{\partial x_2}\right] - \frac{\partial < u_i u_k u_2 >}{\partial x_2} - \epsilon_{ik} \tag{102}$$

where $U_i = U\delta_{i1}$. Thus viscosity does not appear directly in any of the single point equations governing this region, nor does it appear in those governing the outer boundary layer.

In spite of the above, however, viscosity can be shown to play a crucial role in at least a portion of the constant stress layer, even at infinite Reynolds number. The reason is that the scales of motion at which the dissipation, ϵ_{ik}, actually takes place depend on the *local* turbulence Reynolds number, $R_t = q^4/\nu\epsilon$. Above $R_t \sim 5000$ approximately[6] These are effectively inviscid, but control the energy transfer through non-linear interactions (the energy cascade) to the much smaller viscous scales where the actual dissipation occurs (v. Tennekes and Lumley 1972). When this is the case, the dissipation is nearly isotropic so $\epsilon_{ik} \approx 2\epsilon\delta_{ik}$. Moreover, ϵ can be approximated by the infinite Reynolds number relation, $\epsilon \sim q^3/L$, where L is a scale characteristic of the energy-containing eddies. The coefficient has a weak Reynolds number dependence, but is asymptotically constant. Thus, the Reynolds stress equations are effectively inviscid, but only exactly so in the limit. And in this limit the Reynolds shear stress has no dissipation at all, i.e., $\epsilon_{12} = 0$. (Note that these are nearly the same conditions required to observe a $k^{-5/3}$-range in the energy spectrum, cf. Batchelor 1953.)

At very low turbulence Reynolds number, however, the dissipative and energy-containing ranges nearly overlap, and so the latter (which also produce the Reynolds shear stress) feel directly the influence of viscosity. In this limit, the energy and dissipative scales are about the same, so the dissipation is more reasonably estimated by $\epsilon \sim \nu q^2/L^2$, where the constant of proportionality is of order 10. The dissipation tensor, ϵ_{ik} is anisotropic and ϵ_{12}, in particular, is non-zero (Launder 1993). (Hanjalic and Launder 1972, for example, take $\epsilon_{12} = - < u_1 u_2 > /q^2\epsilon$.)

For turbulence Reynolds numbers between these two limits, the dissipation will show characteristics of both limits, gradually making a transition from $\epsilon \sim \nu q^2/L^2$ to $\epsilon \sim q^3/L$ as R_t increases. Thus the Reynolds stresses themselves will feel this directly through their balance equations, and will consequently show a Reynolds number dependence. Obviously, in order to establish when (if at all) parts of the flow become Reynolds number independent, it is necessary to determine how the local turbulence Reynolds number varies downstream and across flow.

Over the outer boundary layer (which is most of it), $L \approx 3\theta$ and $q \approx 0.1U_\infty$. So when $U_\infty\theta/\nu > 15,000$, the dissipation in the outer flow is effectively inviscid. Alternatively, $L \approx 0.3\delta$ and $q \approx 2.5u_*$ so this corresponds to $\delta^+ > 5,000$. Above these values the mean and turbulence quantities in the outer flow should show little Reynolds number dependence, and this is indeed the case — when they are scaled properly! This outer region can, of course, not be entirely Reynolds number independent, except in the limit, and this residual dependence manifests itself in the overlap layer in the slow variation of γ, for example.

[6] This insures that the energy-containing scales of motion, say L, are about 20 times larger than the Kolmogorov microscale, η_K since $L/\eta_K = (R_t/9)^{3/4}$, the energy dissipation is nearly completely controlled by the large energetic scales of motion. The peak in the dissipation spectrum is at about $6\eta_K$, and most of the dissipation occurs at smaller scales (Tennekes and Lumley 1972). As pointed out by a reviewer, this also insures less than a 1% contribution to the turbulent diffusion.

The near wall region is considerably more interesting since in it the scales governing the energy-containing eddies are constrained by the proximity of the wall. Hence, the turbulence Reynolds number, R_t, depends on the distance from the wall, y. Using $L \sim y$ and $q \sim 0.4u_*$ yields $R_t \approx 18y^+$; so, in effect, y^+ is the turbulence Reynolds number. Because of this, two things are immediately obvious:

- First, since a fixed value of y^+ does not move away from the wall as fast as δ, then as the Reynolds number increases more and more of the boundary layer (in outer variables) will become effectively inviscid and will be governed by the inviscid dissipation relation. And correspondingly, the mean and turbulence quantities in the overlap layer will become Reynolds number independent, albeit very slowly. Clearly this limiting behavior cannot be reached until at least part of overlap layer, say the inertial sublayer, is governed by the infinite Reynolds number dissipation relation and its coefficient has reached the limiting value. Obviously this can happen only when there is a substantial "*inertial subrange*" satisfying $y^+ > 300$ and for which the mean convection terms are negligible, typically $\overline{y} < 0.1$. Thus the asymptotic limits are realized only when $300\nu/u_* << 0.1\delta$ or $u_*\delta/\nu >> 3000$, which corresponds approximately to $U_\infty\theta/\nu >> 10,000$. Note that a choice of $\overline{y} < 0.15$ would bring these numbers into coincidence with those for the outer flow, but at most all of these choices are approximate. Regardless, all estimates for where inertial effects dominate the dissipation and Reynolds stress in the near wall region are near the highest range of the available data which end at about $50,000$. Therefore the inertial sublayer, to the extent that it is identifiable at all, should (and does, as will be seen in Part II) display a Reynolds number dependence, not only in C_o, C_i, and γ, but correspondingly in the behavior of $< u^2 >$, $< uv >$, etc.

- Second, there will always be a region, hereafter referred to as the *MESOLAYER*, [7] below about $y^+ \approx 300$ in which the dissipation (and Reynolds stress) can *never* assume the character of a high Reynolds number flow, no matter how high the Reynolds number for the boundary layer becomes. This is because the dissipation (and Reynolds stress) can never become independent of viscosity — even though the mean momentum equation itself is inviscid above $y^+ \approx 30$! This fact is well-known to turbulence modellers (v. Hanjalic and Launder 1972), but the consequences for similarity theory and asymptotic analyses do not seem to have been noticed previously. It is particularly important for experimentalists who have routinely tried to apply asymptotic formulas to this region, wrongly believing the mesolayer to be the overlap region.

Thus the constant stress layer is really two separate regions, an overlap region and a viscous sublayer, each having two subregions. The overlap region obtained in the preceding section consists of an inertial sublayer ($y^+ > 300$, $\overline{y} < 0.1$) which is nearly inviscid; and a 'mesolayer' ($30 < y^+ < 300$) in which the viscous stresses are negligible, but in which viscosity acts directly on the turbulence scales producing the Reynolds stresses. The viscous sublayer consists of a buffer layer ($5 < y^+ < 30$) where the Reynolds stress and viscous stress both act directly on the mean flow; and the linear sublayer near the wall ($y^+ < 3$) where the viscous stresses dominate. And of these four sub-regions, the inertial sublayer will be the *last* to appear as the flow develops or as the Reynolds number is increased. Thus it will be the most difficult to identify at the modest Reynolds numbers of laboratory experiments. Identification will be easier if the properties of the mesolayer are known, and accordingly a model for it is presented in the next section.

[7] This appropriates a term from Long 1976 (see also Long and Chen 1982) who argued strongly for its existence, but from entirely different physical and scaling arguments which we find untenable. Nonetheless, despite the skepticism which greeted his ideas, Long's instincts were correct.

12 A Mesolayer Interpretation of a^+

The role of the parameter a can be better understood by expanding the inner velocity profile of equation 81 for $y^+ >> a^+$. The result is

$$\frac{U}{u_*} = C_i {y^+}^\gamma + \gamma C_i a^+ {y^+}^{\gamma-1} + \frac{1}{2}\gamma(\gamma-1){a^+}^2 {y^+}^{\gamma-2} + \ldots \tag{103}$$

For $y^+ >> 2|a^+/(1-\gamma)|$, this can be approximated by the first two terms as

$$\frac{U}{u_*} = C_i {y^+}^\gamma + \gamma C_i a^+ {y^+}^{\gamma-1} \tag{104}$$

It appears from the data that $a^+ \approx -37$, so that the neglected term is less than 20% for $y^+ > 100$.

This form is particularly useful for three reasons: First, it is an excellent approximation equation 81 for all values of $y^+ > -a^+$. Second, it is easier to incorporate into a composite solution for the inner region since it does not have the singularity at $y^+ = -a^+$. Third, it is very close to the mesolayer/overlap solution derived in Appendix III from a single equation turbulence model; i.e.,

$$u^+ = C_i {y^+}^\gamma + C_{mi}/y^+ \tag{105}$$

For the data shown in Part II, $C_{mi} \approx -37$. The turbulence model essentially regards the dissipation in the near wall region to be a linear combination of a high Reynolds number dissipation ($\epsilon \sim q^3/y$) and a low Reynolds number dissipation contribution ($\epsilon \sim \nu q^2/y^2$). The first term in equation 105 arises from the high Reynolds number part, and thus describes the region where the Reynolds stress and dissipation is nearly inviscid; i.e., above $y^+ > 300$. It is clearly responsible for the power law part of the profile. The second term in equation 105 arises from the low Reynolds number contribution to the dissipation, and only contributes significantly for $y^+ < 300$.

Considerable insight into the role of the a^+ (or $\overline{a}$) in the overlap solution cam be obtained by comparing the terms of equations 104 and 105. Clearly the two expressions are identical for large values of y^+ since the first term dominates. This is the inertial sublayer portion of the profile expected from the arguments above. The second terms differ only slightly in the exponent of y^+ (and not at all for the pipe and channel flows considered in the appendix)[8]. Obviously this is the mesolayer correction which remarkably is included in the Near-Asymptotics result of equation 81 with no model at all. By ignoring the slight difference in exponent, an approximate relation between a^+ and C_{mi} can be obtained as

$$C_{mi} \approx \gamma C_i a^+ \tag{106}$$

or

$$a^+ \approx \frac{C_{mi}}{\gamma C_i} \tag{107}$$

For the range of Reynolds number of the boundary layer data considered in Part II, $a^+ \approx C_{mi}$ since $\gamma C_i \approx 1$. The mesolayer parameter, C_{mi} was found to be constant at approximately -37 so $a^+ \approx -37$ also.

Because of the difficulties presented by the singularity of equations 80 and refeq:powerin at $y = a$, only the expanded forms of equations 104 or 105 will be considered later. These forms will be seen to be considerably

[8] It might be interesting to explore why the turbulence model and overlap theory differ for one flow and not the other.

easier to build composite solutions from as well which include the buffer and linear regions as well. Note that equation 104 can be written in outer variables

$$\frac{U}{U_\infty} = C_o\overline{y}^\gamma + \gamma\overline{a}C_o\overline{y}^{\gamma-1} \tag{108}$$

where $\overline{a} = a^+/y^+_{1/2}$.

13 The Reynolds Stress in the Overlap Layer

By following the same procedure as for the velocity, the outer and inner Reynolds stress profile functions for the overlap region can be obtained. For example, the Reynolds shear stress is given by,

$$r_o(\overline{y};\delta^+) = D_o(\delta^+)(\overline{y}+\overline{a})^{\beta(\delta^+)} \tag{109}$$

$$r_i(y^+;\delta^+) = D_i(\delta^+)(y^+ + a^+)^{\beta(\delta^+)} \tag{110}$$

where a solution is possible only if

$$\frac{R_{so}}{R_{si}} = \frac{D_i}{D_o}\delta^{+\beta} \tag{111}$$

and

$$ln\delta^+ \frac{d\beta}{d\ln\delta^+} = \frac{d}{d\ln\delta^+} ln\left[\frac{D_o}{D_i}\right] \tag{112}$$

Note that the last equation must also be invariant to transformations of the form $\delta \rightarrow D\delta$, just as for the corresponding velocity constraint. It has been assumed that the scale factor D and the origin shift represented by a is the same as for the velocity, since any other choice does not seem to make sense physically.

Unlike the velocity, however, more information about the Reynolds stress is available from the averaged momentum equation for the overlap layer since both equations 2 and 3 reduce to

$$\frac{\partial}{\partial y}(-<uv>) = 0 \tag{113}$$

in the limit of infinite Reynolds number. Thus,

$$\beta R_{so}D_o\overline{y} + \overline{a}^{\beta-1} \rightarrow 0 \tag{114}$$

and

$$\beta R_{si}D_i y^+ + a^{+\beta-1} \rightarrow 0 \tag{115}$$

Since both D_o and D_i must remain finite and be asymptotically constant (if the Reynolds stress itself is non-zero), these conditions can be met only if

$$\beta \rightarrow 0 \tag{116}$$

From equation 30 for large values of y^+, the Reynolds stress in inner variables in the matched layer is given to first order (exact in the limit) by

$$r_i \rightarrow 1 \tag{117}$$

Since $R_{si} = u_*^2$, this can be consistent with equation 110 only if $D_i \to 1$ as $\delta^+ \to \infty$. It follows immediately that

$$R_{so} \to \frac{D_i}{D_o} u_*^2 \tag{118}$$

in the infinite Reynolds number limit, just as suggested in Section 7.

Some insight into the behavior of $D_o(\delta^+)$ and $D_i(\delta^+)$ can be obtained by introducing the momentum integral equation defined by

$$\frac{d\theta}{dx} = \frac{u_*^2}{U_\infty^2} \tag{119}$$

Using this, equation 118 and the similarity relation for R_{so} from equation 37 yields

$$\frac{D_o(\delta^+)}{D_i(\delta^+)} = \frac{d\theta/dx}{d\delta/dx} \tag{120}$$

The relationship between θ and δ will be explored in more detail below, and it will be shown that θ/δ is asymptotically constant. Thus the scale for the outer Reynolds stress is asymptotically proportional to u_*^2 as noted earlier, and the outer layer is indeed governed by two velocity scales. Note that for *finite* Reynolds numbers, both D_o and D_i are Reynolds number dependent. Hence, u_*^2 *alone* should not be able to perfectly collapse the Reynolds stress in either the overlap or outer layers, except possibly in the limit of infinite Reynolds number. This has been observed by numerous experimenters (e.g., Klewicki and Falco 1993) who show persistent Reynolds number trends in the Reynolds stress measurements.

The interrelation of the Reynolds stress and velocity parameters can be examined by considering the production term $< -uv > \partial U/\partial y$. Since this must be the same whether expressed in inner or outer variables, it follows that

$$C_o D_o U_\infty^2 \frac{d\delta}{dx} \sim C_i D_i u_*^2 \tag{121}$$

or asymptotically, $D_o = C_i(D_i/C_o)$. The experiments considered in Part II show that C_o and D_i achieve a nearly constant value for relatively low values of R_θ, while C_i only approaches a constant value for much higher Reynolds numbers. Obviously the outer Reynolds stress parameter, D_o, follows the inner velocity, thus emphasizing the role of the boundary condition provided by the Reynolds stress on the outer flow by the inner.

14 A Composite Velocity Profile

It is possible to use the information obtained in the preceding sections to form a composite velocity profile which is valid over the entire boundary layer. This is accomplished by expressing the inner profile in outer variables, adding it to the outer profile and subtracting the common part (Van Dyke 1964), which is the profile for the overlap region. Alternatively, the outer profile could be expressed in inner variables, etc.

The composite velocity profile in outer variables is given by

$$\frac{U}{U_\infty} = \left[1 + f_o(\overline{y}, \delta^+)\right] + \frac{u_*}{U_\infty}\left[f_i(\overline{y}\delta^+, \delta^+) - C_i\left(\overline{y}\delta^+\right)^\gamma\right] \tag{122}$$

Recall that f_o, f_i, C_i and γ are all functions of δ^+, as is u_*/U_∞. The mesolayer contribution has been considered to be part of the inner solution, but could have been included with the common part since it is known.

The composite velocity solution has the following properties:

- As $\delta^+ = \delta/\eta \to \infty$, for finite values of $\overline{y}$, $U/U_\infty \to 1 + f_{o\infty}(\overline{y})$. Thus there is a boundary layer profile even in the limit of infinite Reynolds number and it corresponds to the outer scaling law. This can be contrasted with the Millikan approach for which $U/U_\infty \to 1$, a limit remarkably like no boundary layer at all, even in its own variables.
- As $\overline{y} \to 0$, $U/U_\infty \to (u_*/U_\infty) f_i(\overline{y}\delta^+, \delta^+)$ for all values of δ^+. This is because the small $\overline{y}$ behavior of $[1 + f_o(\overline{y}, \delta^+)]$ is cancelled out by the last term leaving only the inner solution.
- As $\overline{y}\delta^+ \to \infty$ for all values of δ^+, $U/U_\infty \to 1 + f_o(\overline{y}, \delta^+)$. This is because the large $\overline{y}\delta^+$ behavior of f_i is cancelled by the last term.
- In the overlap region, only the power law profile remains.

It is an interesting exercise to substitute the composite solution into the full boundary layer equation given by equation 1. As expected, it reduces to equation 2 for infinite Reynolds number and to equation 3 as the wall is approached. This can be contrasted with the substitution of the Millikan/Clauser log law plus wake function (v. Coles 1956) in which the outer equation vanishes identically in the limit of infinite Reynolds number.

An alternative composite solution can be obtained by multiplying the inner and outer solutions together and dividing by the common part; i.e.,

$$\frac{U}{U_\infty} = \frac{[1 + f_o(\overline{y}, \delta^+)] f_i(\overline{y}\delta^+, \delta^+)}{C_o \overline{y}^\gamma} \tag{123}$$

For the zero pressure-gradient boundary layer, this composite solution is nearly indistinguishable from equation 122 when plotted against the experimental data.

15 The Displacement and Momentum Thicknesses

The displacement thickness, δ_*, is defined by

$$U_\infty \delta_* \equiv \int_0^\infty (U_\infty - U) dy \tag{124}$$

This can be expressed using equation 122 as

$$\frac{\delta_*}{\delta} = -I_1 - I_2 {R_\delta}^{-1} \tag{125}$$

or

$$\frac{\delta}{\delta_*} = -\frac{1}{I_1}(1 + \frac{I_2}{R_{\delta_*}}) \tag{126}$$

where

$$I_1 \equiv \int_0^\infty f_o(\overline{y}, \delta^+) d\overline{y} \tag{127}$$

$$I_2 \equiv \int_0^\infty \left[f_i(y^+, \delta^+) - C_i {y^+}^\gamma \right] dy^+ \tag{128}$$

and the Reynolds numbers R_δ and R_{δ_*} are defined by

$$R_\delta = \frac{U_\infty \delta}{\nu} \tag{129}$$

and

$$R_{\delta_*} = \frac{U_\infty \delta_*}{\nu} \tag{130}$$

The integrals I_1 and I_2 are functions only of the Reynolds number and become asymptotically constant.

The momentum thickness, θ, is defined by

$$U_\infty^2 \theta \equiv \int_0^\infty U(U_\infty - U) dy \tag{131}$$

Again using equation 122, the result is

$$\frac{\theta}{\delta} = -(I_1 + I_3) - {R_\delta}^{-1} \left[I_2 + 2I_4 + I_5 \frac{u_*}{U_\infty} \right] \tag{132}$$

or

$$\frac{\delta}{\theta} = -\frac{1}{I_1 + I_3} \left\{ 1 + R_\theta^{-1} \left[I_2 + 2I_4 + I_5 \frac{u_*}{U_\infty} \right] \right\} \tag{133}$$

where

$$R_\theta = \frac{U_\infty \theta}{\nu} \tag{134}$$

and

$$I_3 \equiv \int_0^\infty \left[f_o(\overline{y}, \delta^+) \right]^2 d\overline{y} \tag{135}$$

$$I_4 \equiv \int_0^\infty \left[f_i(y^+, \delta^+) - C_i {y^+}^\gamma \right] f_o(y^+/\delta^+, \delta^+) dy^+ \tag{136}$$

$$I_5 \equiv \int_0^\infty \left[f_i(y^+, \delta^+) - C_i {y^+}^\gamma \right]^2 dy^+ \tag{137}$$

Since u_*/U_∞ varies in the limit as $(U_\infty \delta/\nu)^{-\gamma/(1+\gamma)}$ and $\gamma \geq 0$, all terms but the first vanish in the limit of infinite Reynolds number. Thus, as for the displacement thickness, the momentum thickness is also asymptotically proportional to the outer length scale, but with a different constant of proportionality. It will be seen in Part II that this limit is approached very slowly, and the limiting value is achieved at Reynolds numbers well above those at which experiments have been performed.

The shape factor can be computed by taking the ratio of equations 125 and 132. The result is

$$\begin{aligned} H &\equiv \delta_*/\theta \\ &= \frac{I_1 + I_2 R_\delta^{-1}}{(I_1 + I_3) + R_\delta^{-1}(I_2 + 2I_4 + I_5 u_*/U_\infty)} \end{aligned} \tag{138}$$

For large values of Reynolds number, the asymptotic shape factor is easily seen to be given by

$$H \rightarrow \frac{I_1}{I_1 + I_3} \tag{139}$$

Note that since $f_o \leq 1$ always, it follows from their definitions that $I_1 < 0$, $I_3 > 0$ and $|I_1| > I_3$. Therefore the asymptotic shape factor is greater than unity, in contrast to the old result, but consistent with all experimental observations.

It is obvious from equations 125 and 132 that both the displacement and momentum boundary layer thicknesses are asymptotically proportional to the outer length scale (or boundary layer thickness) used in the analysis. Note that it does not matter precisely how this outer length scale is determined experimentally, as long as the choice is consistent and depends on the velocity profile in the outer region of the flow (e.g., $\delta_{0.99}$ or $\delta_{0.95}$). This is quite different from the Millikan/Clauser theory (with finite κ) where the displacement and momentum thicknesses vanish relative to the unspecified outer length scale.

16 Streamwise Dependence of the Boundary Layer

The friction coefficient can be written entirely in terms of R_θ by using equations 133 and 100; i.e.,

$$c_f = 2(\frac{C_o}{C_i})^{2/(1+\gamma)} \Big[\left(\frac{-1}{I_1 + I_3} \right) (R_\theta + I_2 + 2I_4) \Big]^{-2\gamma/(1+\gamma)} \tag{140}$$

where the term involving $I_5 u_*/U_\infty$ has been neglected. Also, I_2 and I_4 are much less than R_θ, so that

$$c_f \approx 2(\frac{C_o}{C_i})^{2/(1+\gamma)} \Big[\left(\frac{-1}{I_1 + I_3} \right) (R_\theta) \Big]^{-2\gamma/(1+\gamma)} \tag{141}$$

This theoretical result provides a point of reference with Schlichting (1968) who noted that a 1/7-power law fit the velocity profile approximately over a limited range of the data. Using $\gamma = 1/7$ leads immediately to $c_f \sim R_\theta^{1/4}$ which was the corresponding friction law. Note that for higher Reynolds numbers Schlichting suggested $c_f \sim R_\theta^{1/5}$ which corresponds to $\gamma = 1/9$, consistent with the idea that γ is indeed Reynolds number dependent. The suggested asymptotic value of $\gamma_\infty = 0.0362$ gives $c_f \sim R_\theta^{0.07}$ as the limiting power law.

The integral of equation 1 across the entire boundary yields the momentum integral equation for a zero-pressure gradient boundary layer as

$$\frac{d\theta}{dx} = \frac{1}{2} c_f \tag{142}$$

Thus the x-dependence of θ can be obtained by integrating

$$\left\{ \left(\frac{-1}{I_1+I_3}\right) \left(\frac{C_i}{C_o}\right)^{1/\gamma} [1 + R_\theta^{-1}(I_2 + 2I_4)] \right\}^{2\gamma/(1+\gamma)} R_\theta^{2\gamma/(1+\gamma)} dR_\theta = dR_x \tag{143}$$

where $R_x \equiv U_\infty x/\nu$.

If the values of C_i, C_o, γ and the I's can be evaluated as functions of R_θ, equation 143 can be integrated numerically to yield the variation of R_θ as a function of $R_x - R_{x_o}$ where R_{x_o} is a virtual origin which will be determined by how the boundary layer is generated. This will be carried out in Part II using empirical relations for the parameters. The x-dependence of H, and the other boundary layer parameters can be similarly determined by substituting the results of the integration into the appropriate equations.

It is interesting to note that if γ can be considered to be constant over some range of Reynolds numbers, then equation 143 can be integrated analytically to obtain

$$R_\theta = \left(\frac{C_o}{C_i}\right)^{2/(1+3\gamma)} \left\{ \left(\frac{1+3\gamma}{1+\gamma}\right) \left[\frac{-1}{I_1+I_3}\right] (R_x - R_{x_l}) \right\}^{(1+\gamma)/(1+3\gamma)} \tag{144}$$

where x_l is the virtual origin for the section of the flow under consideration. Thus the boundary layer thickness is proportional to $x^{(1+\gamma)/(1+3\gamma)}$. For example, if $\gamma \rightarrow 1/10$, $\theta \rightarrow x^{11/14}$. The suggested infinite Reynolds number limit for γ corresponds to $\delta \sim x^{0.935}$. Because of the slow approach of γ to its limiting value, a most important experimental clue that the present analysis is correct will be whether the exponent increases toward unity (or some limiting value close to it, like 0.935) as data points are added from distances farther from the leading edge, especially if data from points close to leading edge are successively dropped.

17 Is the Power Law the Same as the Log Law?

The asymptotic approach of γ to a small value makes it possible to approximately recover the logarithmic relations of the classical theory. The profiles of equations 80 and 81 can be expanded for small values of γ as

$$\frac{U}{u_*} = f_i(y^+; \delta^+) = C_i e^{\gamma ln y^+} \approx C_i(1 + \gamma ln y^+ + \ldots) \tag{145}$$

and

$$\frac{U}{U_\infty} = 1 + f_o(\overline{y}; \delta^+) = C_o e^{\gamma ln \overline{y}} \approx C_o(1 + \gamma ln \overline{y} + \ldots) \tag{146}$$

Thus the asymptotic boundary layer profiles would appear logarithmic to leading order, even for finite values of γ. (The authors are grateful to Prof. Prosperetti of Johns Hopkins University for pointing this out.)

From these "log" profiles and the asymptotic friction value of equation 84 it follows that the effective von Karman/Millikan "constants" of equations 14 and 15 are given by

$$1/\kappa \equiv \gamma_\infty C_{i\infty} \tag{147}$$

$$B_i \equiv C_{i\infty} \tag{148}$$

and

$$B_o \equiv C_{i\infty}(1 - \frac{1}{C_{o\infty}}) \tag{149}$$

Thus the Millikan/Clauser log profile result is recovered as the first term in an expansion.

It is easy to see why the mean velocity profile could have been accepted for so long by experimentalists as logarithmic, since it is very difficult to tell a logarithm from a weak power using experimental data alone since one can always be expanded in terms of the other. Suppose for the moment that it is indeed the overlap region which is being examined and that the present theory is correct, but that an experimenter believed the log theory to be correct with a constant and finite value of κ. The values for C_i and γ at $R_\theta = 50,000$, the limits of experimental data, will be estimated later in Part II to be about 12 and 0.09 respectively. These yield a value of $1/\kappa = 1.1$ which is nowhere close to the generally accepted value of about 2.5, believed to be the asymptotic value for both boundary layers and pipes. Over the range of most experiments, however, $R_\theta \sim 10^4$ and $\gamma \approx 1/7$ while $C_i \approx 10$, which yields about an estimate of $1/\kappa \approx 1.7$. However, the logarithmic expansion converges rather slowly and terms above first order are not negligible (nor were they in the calculation above). To third order in $\gamma \ln y^+$, the effective value of κ is given by

$$\frac{1}{\kappa} \approx \gamma C_i [1 + \gamma \ln y^+ + \frac{1}{2}(\gamma \ln y^+)^2 + \frac{1}{6}(\gamma \ln y^+)^3] \tag{150}$$

Now the presence of y^+ in this expansion is interesting since it is well-known that attempts to fit the log law at modest Reynolds numbers depend on where the point of tangency is chosen. If κ is evaluated by fitting a log profile which is tangent to the data at $y^+ = 100$ (as suggested by Bradshaw 1993), using the above values in the expansion yields an estimate of $1/\kappa \approx 2.36$ or $\kappa \approx 0.42$, which is the value usually assumed. Most often in practice, the experimenter picks his point to obtain the "right" value of κ (hence its *universal* value), and accepts whatever value of the other universal (but highly variable!) *constant* which comes out. In view of this and equation 150, the seemingly paradoxical variability of B_i and constancy of κ is not at all surprising since the 'right' point of tangency can always be found.

As noted above, there is considerable debate in the literature about the value of B_i with cited values ranging from 4 to 12 From equation 148 it is equal to C_i. C_i will be seen later to vary from about 7 to 10 over the range of the low Reynolds number experiments. This is higher than the value of 4.9 suggested by Coles 1968, but well in the range of recent experiments, some of which also show much higher values and a Reynolds number dependence (Nagib and Hites 1995). There is no consensus value for the outer 'constant' B_o, and it is seldom reported at all. Perhaps equation 149 suggests a reason for this in that it is quite small since C_o is never very far from unity. Hence estimates for it would vary widely since the errors might be larger than its value.

In summary, at least some of the general satisfaction (and dissatisfaction as well) with the log law over the range of most experiments can be explained with the new theory. Even the sensitivity noted by experimentalists to the choice of the point of tangency can be explained because of the $\ln y^+$-terms in the expansion of κ for finite values of γ. The power law profiles (and the parameters in them) resulting from the present theory, if correct, should be much less sensitive to the actual range of the data used, especially if the limits imposed by the mesolayer are honored.

It must be remembered that the considerations above apply only to boundary layers and other developing flows, and not to homogeneous wall-bounded flows like pipes and channels. As Appendix I makes clear, these homogeneous flows are indeed described theoretically by logarithmic profiles. These naturally occuring log profiles could of course be expanded approximately as power laws, but with all the problems of Reynolds number dependencies and tangency points noted above. This undoubtedly explains the success of Barenblatt et al. (1997) in fitting power laws to data of Nikuradse (1932), as well as their difficulty in extending their results to the much higher Reynolds numbers of the Zagarola and Smits (1996) data.

Before leaving this section it is interesting to consider one aspect of a finite and non-zero limit for u_*/U_∞, however unlikely it may be in view of the requirement for finite *local* energy dissipation rate discussed earlier. If both u_* and U_∞ were the same in the limit, shouldn't an asymptotic theory based on either alone (like the Millikan/Clauser theory) be correct? An asymptotic approach of γ to zero indeed makes the expansions above exact, and the definitions of equations 147 and 148 must be exactly the Millikan/Clauser constants. The problem is that in the limit as $R_\theta \to \infty$, κ will blow up if $\gamma \to 0$ since $C_{i\infty}$ must be constant (to satisfy the requirements for similarity of the mean momentum equation).

This unseemly behavior of κ is not just a consequence of the theory here but can readily be seen from the old log friction law of equation 16 by requiring that u_*/U_∞ be a non-zero constant. In fact, if the Millikan/Clauser scaling arguments are applied to the turbulence moment equations, then it is easy to show that similarity of the Reynolds stress equations is possible only if $d\delta/dx = constant$ and $u_*/U_\infty = constant$, consistent with the analysis presented herein. And the only possibility of satisfying equation 16 with a finite value of u_*/U_∞ is for κ to increase without bound, exactly as derived here. Therefore, either the old theory is not the limit of the new (if $u_*/U_\infty \to 0$), or it is but with an infinite von Karman constant (if the limiting value of u_*/U_∞ is finite). Obviously the latter possibility makes little sense. Note that precisely the same arguments lead to a *finite* limiting value of the von Karman constant for pipe and channel flows (see Appendix I). Thus the principles are the same; only the results are different, and this is clearly because one flow is developing in x while the other is homogeneous.

Part II
Experimental Data

18 Experimental Overview

Before examining the experimental data for the zero-pressure gradient boundary layer, it is useful to review what should be expected from it. In view of the difficulty of carrying out the experiments, especially close to the wall and in determining the shear stress, it is unlikely that the data will speak unequivocally. Moreover, the experimental problem is complicated by the fact that the all important ratio, u_*/U_∞, decreases ever more slowly with increasing Reynolds number, regardless of what its limiting value is. Therefore experiments at higher Reynolds numbers will be primarily useful for sorting out what the limits are, and less useful for sorting differences in scaling. Experiments at relatively low Reynolds number will be more useful for the latter because the greater variation of u_*/U_∞ will make trends more evident. But even the low Reynolds number experiments will be of little value for establishing the scaling laws unless the shear stress is directly determined, and to an accuracy greater than the variation among experiments. Few experiments to-date satisfy these criteria.

The fact that there are now two competing theories for the boundary layer makes a considerable difference because the data can usefully sort between them. With only one theory, the experimentalist (who is really at his best when sorting theories) is under considerable pressure to obtain results which confirm it, particularly if the theory has been believed to be correct. This has certainly been the case with the "log law" and its consequences. There is ample evidence in the abundant literature of "creative" data analysis to confirm the log results, especially with regard to choice of the appropriate value of the wall shear stress (e.g., Coles 1968, Fernholz et al. 1995). Coles (1968), for example, not only uses equation 14 to determine the shear stress from the velocity profile data, but then uses equation 16 to determine the appropriate value of the boundary layer thickness δ. Thus both of these important parameters are allowed to "float", a reasonable approach only if the theory is known to be true and the constants universal. The fact that few experiments using the values determined in this manner satisfy the momentum integral equation is disturbing, to say the least.

Thus one of the major problems of the present study has been to recognize when data "contamination" has occurred, to attempt to "decontaminate" the data when possible, and to abandon it when that proved impossible. Because a certain element of subjectivity is necessary in this process, a careful attempt has been made to identify precisely what was done so that the reader can judge for himself whether the results here are reasonable. Final judgment can only await a new round of experiments in which the experimentalist is unconstrained by the need to agree with either theory. Fortunately a number of such experiments are in progress. (On-going experiments at Princeton, IIT/Chicago, NASA/Langley, and KTH/Stockholm are known to these authors.)

The task here is somewhat more complicated than simply showing that the new theory is consistent with the experimental data. Since there is an existing theory which has been more or less accepted as being correct, it is also necessary to demonstrate that the data are inconsistent with it. Or failing that, that the new theory works better. Fortunately the recent review of Gad-el-Hak and Bandyopadhyay 1994, in particular, considerably simplifies this endeavor. Their paper is a comprehensive evaluation of the experimental contributions over the past 30 years, and conclusively demonstrates the inadequacies of the von Karman/Millikan scaling laws. In particular, they document the failure to achieve Reynolds number independence of either the outer scaling for the mean velocity deficit, or of the turbulence quantities outside of the linear sublayer. Therefore their effort will not be repeated here, and attention will be focused on showing that the data are at least as consistent with the new theory as with the old. In fact, it will be possible to show that some of the alleged 'problems' with the data disappear when viewed from the perspective of the new theory .

19 The Velocity Deficit Region

Since there is usually little doubt as to the proper value of U_∞ in any given experiment, there is no data manipulation which can be done, particularly if a directly determined length scale (e.g., like δ_{99}) is used. Unfortunately, as will be seen below, there can be considerable doubt as to the value of u_*, and this presents serious problems to any evaluation of the data. The question of what is the proper value of u_* will be deferred until the next section, except to note that it will often be seen to be in dispute.

Figure 1 shows plots of the data of Purtell et al. 1981 ($465 < R_\theta < 5,100$) in the outer variables of equations 9 and 10 in both linear-linear (leftmost) and semilog (rightmost) plots. For the U_∞-scaling shown at the topmost, the collapse is excellent far away from the wall at all Reynolds numbers, and the point of departure from the "asymptote" moves closer to the wall with increasing Reynolds numbers. This is the expected behavior for an asymptotic outer solution. However, when the Purtell et al. data are plotted in the usual outer variables using u_* determined from the velocity gradient near the wall (bottom figures), there is an obvious tendency for the collapse to come apart away from the single point where it must collapse because of the outside boundary condition. That this happens in the heart of the outer flow where an outer scaling law should work best is at the very least disconcerting from the perspective of the old theory. Absent almost completely is the expected splitting off toward the wall with increasing Reynolds number. The middle figures of Figure 1 show the same data, but using values of u_* chosen to collapse the log layer. (These are in fact the values cited in the paper of Purtell et al.) Even though the data is forced to match at a single point in the "log" region, the same trends are clearly present.

Figure 2 shows the velocity data of Smith and Walker 1959 ($3000 < R_\theta < 50,000$) plotted in deficit form. The shear stresses used in the bottom figures were determined to make the "log" region collapse (the Clauser method discussed below) so that the collapse in the traditional scaling is "optimized". Moreover, the variation of u_*/U_∞ is much less than for the Purtell et al. data because of the higher Reynolds number range. Clauser 1954 plotted only the highest and lowest Reynolds number data of the somewhat less extensive data of Schultz-Grunow 1941 for the U_∞ scaling and concluded the absence of collapse ruled it out as as suitable scaling law. The u_*-scaled data were plotted only as a shaded area and deemed acceptable. Certainly if it were expected that the entire velocity profile should be Reynolds number independent, then Clauser's conclusion is reasonable. On the other hand, if it had been expected that the outer layer is only asymptotically independent of Reynolds number, then the U_∞ scaling might have merited further consideration, particularly since as the Reynolds number increases the remainder of the Schultz-Grunow profiles, like the Smith/Walker profiles, are clearly moving toward that of the highest Reynolds number, and the region of collapse is moving toward the wall.

That the old outer scaling in previous works was deemed acceptable is probably due to the moderate Reynolds number range of the data, the choice of u_* to force the "log region" to overlap, the indirect determination of δ, and the mixed (inner and outer) nature of the variables used (i.e., u_* and δ). In spite of this, the outer data normalized by u_* do not appear to be consistent with the asymptotic nature of an outer scaling law. This is closely related to the observations of Gad-el-Hak and Bandyopadhyay 1994 who noted the failure of the "law of the wake " to reach an asymptotic state.

Thus, contrary to previous interpretations, the data would seem to indicate a preference for the new formulation of the velocity deficit law using U_∞ (equation 9). Further evidence for the new deficit law is shown in Figure 3 using the recent data of Klewicki 1989. The more limited Reynolds number range of this data ($1,000 < R_\theta < 5,000$) makes the Reynolds number independent range appear to be even larger than in the other two data sets. It was noted earlier that at these low Reynolds numbers u_*/U_∞ is varying quite rapidly so the success of the U_∞-scaling is quite impressive. Smith (1994) in one of the first experiments designed in part to test the theory described herein found similar success for scaling the velocity deficit with U_∞ from $5,000 < R_\theta < 13,000$.

20 The Near Wall Region

The near wall region can not be addressed without first deciding what values of u_* to use for a given data set. Unfortunately, direct measurements of τ_w have, for the most part, not been very satisfactory for a variety of reasons, not the least of which being that the results did not confirm the Millikan/Clauser theory. Therefore it has been common practice (since Clauser 1954) to choose values for u_* which are consistent with the "universal" log law results for the matched layer, by which is usually meant the constants chosen from the more abundant pipe and channel flow data. Coles (1968) notes that the results obtained in this way are seldom consistent with the momentum integral equation. Nonetheless, in spite of the fact the measurement errors tend to cancel when the integrals are computed from the velocity profiles and therefore should be more accurate than the profiles themselves, the friction results calculated from them have been usually discarded in favor of the Clauser method. Typical of such results is Figure 4 which shows the data of Purtell et al. 1981 in inner variables using the wall stress determined from the Clauser method (bottommost) and from the velocity gradient near the wall (topmost). Note the excellent collapse in the "log" region of the former, but also recognize that the data should collapse best in this region because u_* has been chosen to insure that it does — in fact the choice of the tangency point for the log is obvious from the figure. More interesting, however, is what has happened to the measurements in the linear layer for small values of $y^+ < 8$ or so where the data do not collapse at all.

This problem with the inner scaling has been noted before (e.g., Kline et al. 1967), and it seems to have been customary to opt for the method which collapses the "log layer", and rationalize the problems this presents by assuming the measurements near the wall to be in error. While certainly all measurement sets are subject to inaccuracies near the wall, the choice represents more an expression of faith in the log law than a consequence of careful error analysis. In fact, curiously, measurements in the linear regions of pipes and channels seem to have been routinely made over the past 50 or so years with little difficulty as long as the ratio of wire diameter to distance to the wall was greater than about 100 (v. Monin and Yaglom 1972, vol. I, figure 25). Perry and Abell 1975 measure to within $0.4mm$ with $4\mu m$ hot-wires in a pipe flow before noticing effects of wall proximity. By the same criterion, the data for Purtell et al. should be expected to be valid to values of y^+ from 2 to 5 for eight of the 11 data sets. Thus, either hot-wires have behaved very differently in boundary layers than in pipes, or the experimenters have been unwilling to accept the answers the wires were providing.

The values of u_* from the velocity gradient were made using a linear fit for the data in the region $3 < y^+ < 6$, and points closer to the wall were ignored since they were clearly contaminated by near wall effects on the wires. (It will be seen below that there is evidence that the shear stress has been underestimated by about 20% by this method, but with relative errors which are nearly Reynolds number independent.) The data show clearly that the linear region both exists and the data in it collapse nearly perfectly. Moreover the extent of the linear region appears to be greater than for pipes and channels, but decreases with increasing Reynolds number. As expected from the new theory, the data outside the linear layer does not perfectly collapse in inner variables, but shows the Reynolds number dependence which must be present if the inner and outer scale velocities are different.

Similar profiles for the near wall region were published by Blackwelder and Haritonidis 1983 who documented the difference between the shear stress inferred from the velocity gradient near the wall and from a fit to the "log" region. These systematic differences were typically 15% and are tabulated in Table I of their paper. Their plot of the velocity profile (normalized using the shear stress determined from the linear region) is strikingly similar to that of the Purtell data plotted in Figure 4. Of special interest is the fact that the very near wall data show clearly the onset of the near wall measurement errors well inside the linear region, except for the highest Reynolds numbers.

In the absence of direct force measurements, the fact that the velocity profile near the wall collapses in both these data sets using the wall shear stress inferred from the gradient suggests strongly that both the mean velocity and the shear stress have been reasonably estimated, at least at the lower Reynolds numbers. A similar inference cannot be made using a value of the stress which collapses a "log" region, since (as pointed out above) it is based on the assumptions that such a region exists and that the constants describing it are Reynolds number independent. Clearly the second of these is incorrect if the profiles normalized using the shear stress inferred from the velocity gradient are correct, and the first may be wrong as well.

The first important inference which can be seen from these figures is that the data collapse (if a cross-over can be called 'collapse') in the "log" region only if the shear stress is calculated from a method which forces it to by assuming such a layer exists, and then only by compromising the collapse in the linear region. On the other hand, when the measured shear stress is used, the profiles collapse well very close to the wall, but not in the “log” layer. The lack of collapse in the linear layer when the “Clauser method” is used is particularly serious since there are no adjustable constants or Reynolds number dependencies here: the measurements must yield $u^+ = y^+$ if they are to be believed.

A second important inference from the figures is that when the data are normalized by a “proper” shear stress, the point of departure from the linear region moves closer to the wall with increasing Reynolds number toward what might be a limiting value. This is clear evidence that if there is an intermediate layer outside the linear region, then it is only asymptotically independent of Reynolds number. The apparent Reynolds number dependence of the extent of the near-linear region is consistent with the new theory presented earlier since only this region of the flow is Reynolds number independent in inner variables. Once the mesolayer and overlap region are entered, neither u_* nor U_∞ completely collapse the data (except in the Infinite Reynolds number limit) until the influence of the inner region has disappeared and the outer flow has been reached. Many (v. Gad-el-Hak and Bandyopadhyay 1994) have noted similar trends for the turbulence moment data near the wall, especially $< u^2 >$ and $< -uv >$.

21 The Overlap Layer and the Mesolayer

Figure 5 shows log-log (leftmost) and semilog (rightmost) plots in inner variables (using the velocity gradient at the wall to obtain the shear stress) for a few of the many cases which were considered. It is easy to argue from the log-log plots that there is evidence in all the data of an extensive power law region ranging from approximately $y^+ > 50$ to $\overline{y} < 0.2$. However, as noted in sections 11 and 29, great care must be used before inferring that this is the overlap region, especially at these low Reynolds numbers. It will be seen later that indeed it is not. The semilog plots show for comparison the same data in the traditional semi-log plot in inner variables along with the the traditional log profile ($1/\kappa = 2.44$, $B = 5.0$). Obviously an argument for a log region can also be made, although perhaps not the traditional one.

Spalart 1988 recognized the difficulty of determining functional dependencies from velocity profile plots and suggested plotting $y^+ du^+/dy^+$ versus y^+ since a constant region of the former would indicate a logarithmic variation of U. In view of the possibility of a power law region, a more useful plot is $y^+ du^+/dy^+$ versus u^+ since it yields a constant value if the relation is logarithmic and a linearly varying region if a power law. (Note that it does not matter if outer variables are used instead of inner.) Figure 6 shows profiles of ydU/dy versus U using the data of of Purtell et al. in both inner and outer variables. While the data are subject to interpretation because of the large scatter, there is some evidence of the power law region in each set, and certainly considerably more evidence than for a log region. Also evident is the Reynolds number dependence of the slope and intercept, consistent with the theory and the observations above.

There are several complicating factors in using any of the type of plots above to determine whether there is a power law region (or for that matter, a log region). The first is that if there is a mesolayer region in which the power law is modified (as argued earlier), then any data inside $y^+ = 300$ approximately must be excluded. This eliminates all of the Purtell et al. data. The second is that the data outside $\overline{y} \approx 0.1$ (or $y^+ \approx 0.1\delta^+$) must be eliminated since the convection terms are becoming important. Thus, none of the velocity data below about $R_\theta = 10,000$ can be used — and that is most of the available data. Fortunately, by constructing a composite solution for the entire profile and applying it to all the data at once, it is possible to solve for the parameters of the individual pieces, even though they describe regions which are not clearly visible since they are simultaneously influenced by different physics.

The parameters C_o, C_i and γ were obtained from the measurements by a variety of composite solutions and optimizations of increasing sophistication. The latter are outlined in Appendix II. The various components of these composite solutions are presented below in Section 23. These composite solutions, together with the empirical functions for the wake and buffer regions (as defined in section 23 below) made it possible to use all of the available measurements (including the integral thicknesses) in the optimization. Only the velocity data of Purtell et al. and Smith/Walker were utilized in these determinations. There was remarkable consistency in the estimates of both γ and C_o for all of the methods utilized, especially when the role of the mesolayer was accounted for. The mesolayer parameter itself was found to be approximately constant at $C_{mi} = -37$.

The values of γ, C_o, C_o/C_i and C_i are plotted in Figure 7 as functions of δ^+. Note for future reference that the relation between δ^+ and R_θ is given by

$$R_\theta = \frac{U_\infty}{u_*}\frac{\theta}{\delta}\delta^+ \tag{151}$$

since $\delta^+ = u_*\delta/\nu$. The conversion from one to the other can either be done using the data, or the theoretical relations for u_*/U_∞ and θ/δ determined below; these will be shown later in Figure 17 after the development of the relationships.

The values of γ shown in Figure 7 drop rapidly over the same range for which C_o shows most of its variation ($R_\theta < 1000$), and then ever more slowly. Whether γ would continue to drop beyond $R_\theta > 50,000$ can not be determined from the data, which are consistent with both non-zero and zero asymptotic values of γ.

The suggestion of Barenblatt (1993) that $\gamma = a/\ln\delta^+$ does provide a reasonable fit to the data for γ; however, the $C_o/C_i = b(e/\ln\delta^+)^a$ — which results from the constraint equation — does not. As noted earlier, this failure can most likely be attributed to the facts that $\gamma \to 0$ is inconsistent with the requirement for finite energy dissipation, and that $C_o/C_i \to 0$ does not satisfy the similarity constraint.

A modified form of Barenblatt's assumption, however, which is consistent with both the Navier-Stokes equations and similarity is

$$\gamma - \gamma_\infty = \frac{-\alpha A}{(\ln\delta^+)^{1+\alpha}} \tag{152}$$

where $\alpha > 0$ is a necessary condition. Of all the forms tried for which solutions to the constraint equation could be found, this provided the best fit to all the parameters, and it was used in the final optimization below.

The parameters, γ, C_o and C_i proved to be strongly interrelated, hence the scatter in the data when each profile was considered individually. Therefore the final determination of the constants was determined by including equation 152 and the corresponding solutions for C_o/C_i and u_*/U_∞, together with all the profiles simultaneously, in the final optimization discussed below. It should be noted that equation 152 is the *only* empirical equation that enters the entire theoretical formulation, aside from the interpolations used to construct the composite profile. Most importantly, it enters at the very end of the analysis, and not at the beginning as in the classical theory (the assumed velocity deficit law).

C_i can be determined either from the inner velocity profile data, or from the friction coefficient by substituting the data for c_f and the values of γ and C_o determined above into equation 99. Unfortunately it is impossible to avoid confronting the problem of the shear stress. As is clear from the discussion in the preceding and following sections, there are serious questions about precisely what c_f should be, and those uncertainties directly affect the plots of the velocity data in inner variables. Two different approaches were used: The first attempted to directly obtain the wall shear stress from the data using either a direct estimation of τ_w from the velocity gradient at the wall (Purtell et al. data), or from the momentum integral (Smith/Walker data). The second, which is presented here, used optimization methods which did not depend on any measurement of the shear stress at all but used the theoretical friction law to determine it. One advantage of this approach is that the friction measurements can be used as an independent measure of the success of the theory, since the choice of parameters is independent of them.

The empirical fit to γ used above, equation 152, is particularly useful since as noted earlier $h(\delta^+)$ can be determined from it. From equation 87 it follows that

$$h = \frac{A}{(\ln \delta^+)^\alpha} + h_\infty \tag{153}$$

from which C_o/C_i is given from equation 86 by

$$\frac{C_o}{C_i} = \frac{C_{o\infty}}{C_{i\infty}} \exp[(1+\alpha)A/(\ln \delta^+)^\alpha] \tag{154}$$

It follows immediately from equation 88 that

$$\frac{u_*}{U_\infty} = \frac{C_{o\infty}}{C_{i\infty}} \delta^{+^{-\gamma_\infty}} \exp[(1+\alpha)A/(\ln \delta^+)^\alpha)] \tag{155}$$

Therefore a final optimization can be (and was) performed using all of the profiles simultaneously together with equations 152 to 155 and the semi-empirical composite profile described in the next section (equation 162). The optimal values obtained for the constants were $C_{o\infty} = 0.897$, $C_{i\infty} = 55$, $\gamma_\infty = 0.0362$, $A = 2.90$, and $\alpha = 0.046$. The actual values of the parameters γ, C_o and C_i and C_o/C_i are shown in Figure 7 by the solid lines.

The theory in Part I states that the asymptotic value of C_o should be constant. This indeed appears to be the case as shown in Figure 7, and the asymptotic value of C_o is reached quite early. The data for C_o are well described over the entire range from $465 < R_\theta < 48,292$ by the expression,

$$\frac{C_o}{C_{o\infty}} = 1 + 0.283 \exp(-0.00598\delta^+) \tag{156}$$

The range over which the exponential decay term is important will be seen later to correspond closely to the range in which the wake function (defined as the outer scaled profile minus the power law) is strongly Reynolds number dependent. This behavior is consistent with the Reynolds number arguments put forth earlier for when the outer flow can begin to be Reynolds number independent.

Combining equation 156 with equation 154 leads immediately to an explicit relation for C_i. (Note that C_o could have been assumed constant for all but the lowest Reynolds numbers without significantly changing the results.) These parameters were used in the remainder of this paper for calculating all the velocity profiles, shear stress and integral parameters presented later.

Figures 8 and 9 show 20 of the Purtell et al. and Smith/Walker profiles in outer variables, together with the composite and mesolayer profile of equation 105 using only the parameters determined above and equations 152 – 156. The two vertical dashed lines on each plot mark the limits of the mesolayer model ($30 < y^+ < 300$). Note that for two lowest Reynolds numbers ($Re = 465$ and $Re = 498$), the entire outer region is contained within the boundaries of the mesolayer, so for no region of the flow can the turbulence be described as high Reynolds number. The outer flow develops quickly, however, for $R_\theta > 1000$, but it is only for $R_\theta > 10,000$ that the overlap region begins to emerge. The overlap region goes from the upper extent of the mesolayer to $\overline{y} = 0.1$ approximately, the latter denoted by the light dotted vertical line on the plots for $R_\theta = 13,037$ and higher. Aside from the Smith/Walker data closest to the wall (which are probably in error since they were taken with Pitot tubes and were not corrected), the theoretical relationship provides an excellent description of the flow from well below the expected limits ($y^+ > 30$) to well outside it for all Reynolds numbers. Especially gratifying is the ability of the mesolayer term to capture the low Reynolds number behavior. The entire sequence of profiles from very low to very high Reynolds number is confirmation of the arguments put forth in Section 11.

In spite of the apparent success outlined above, there is still some reason for concern. The Smith/Walker mean velocity data was taken by Pitot tubes for which the measured mean velocity is known to be in error due to turbulence and velocity gradient effects. While hot-wires have a smaller error due to the fluctuating cross-flow, they have other problems which become important as the wall is approached (particularly with calibration at low velocities and heat loss to the wall). In addition, as noted in Part I, a residual dependence of the flow on upstream conditions can not be ruled out from the Reynolds averaged equations. In view of this, the actual values reported need to be refined by further experimentation which takes account of all the factors influencing their determination.

22 The Data for the Friction Coefficient

The theoretical friction law is uniquely determined by the ratio C_o/C_i and by γ. But these are all determined by equation 88 once the function h is specified. As noted above, there is considerable confusion and disagreement about the shear stress, so in the final optimization above, none of the data were considered reliable and only the theoretical values were used. Thus the theoretical friction law depends only on the empirical choice of the $h - function$ and the parameters determined from the velocity profiles.

Figure 10 shows u_*/U_∞ versus R_θ in linear–linear (leftmost) and semilog (rightmost) plots. The theoretical results using equations 151, 155, and the constants determined above are shown by the solid lines. The experimental data in the topmost plots are from the floating element data of Smith/Walker 1959, the force balance data of Schultz-Grunow 1941, and the shear stress computed from the momentum integral data of Wieghardt and Tillman (1944) by Coles (1968). In addition, the data obtained by Spalart (1988) from a DNS simulation of boundary layers at low Reynolds numbers is also included. The agreement between the theoretical result and the experiments is especially remarkable since *NONE* of these data were used in the determination of the empirical constants. Interestingly, both sets of data using direct measurement have previously been widely disregarded since they did not agree with the results obtained by applying the Clauser method to the velocity profile data of the same experiments.

The bottommost figures of Figure 10 show several other data sets which are more or less in agreement with the theory. There is considerable scatter in the results, especially when the velocity gradient estimates from the Purtell et al. data and the Blackwelder/Haridonitis data are considered which are consistently about 15% to 20% low. These low values are due to using values of the velocity to estimate the gradient above $y^+ = 3$ where the Reynolds stress is already beginning to rapidly appear. Similar behavior of plane wall jet data has been noted by Abrahamsson (1996) using the velocity profile data of Karlsson et al. (1993). These data were of sufficient quality that the differences in shear stress estimates from the gradient could be expressly accounted for by using a Taylor expansion for the velocity at the wall (see below), proving that the percent error was indeed Reynolds number independent.

Particular attention must be paid to the extensive data of Smith/Walker, both because of its quantity and the fact that three different methods were used — direct force measurements using a floating element, calculation from the momentum integral using $d\theta/dx$, and the Clauser method — and, perhaps as important, the data have been conveniently tabulated in the report so there is no uncertainty as to what they really are. The c_f deduced from the Smith/Walker momentum thickness data represents $2d\theta/dx$ (actually $2dR_\theta/dR_x$) obtained from fits to the measured values of R_θ and R_x. The values shown are not those presented by Smith/Walker who fitted a curve to $logR_\theta/logR_x$ as a polynomial in R_x, and obtained results closer to the Clauser method than to their floating element results for the lowest Reynolds numbers. The values shown were obtained by fitting $R_\theta/R_x^{1.167}$ as a polynomial in R_x. (The factor of 1.167 was chosen to correspond approximately to the value of γ over the range of the data and the drag law which would result from it using equation 143.) Both of these relations predict nearly the same values of c_f over the highest range of the data, but they diverge for smaller values of R_θ. There is a substantial difference in the extrapolations of these relations to values of $R_\theta < 10^4$: The log-derived curves bend up rather rapidly like the bulk of the data reported using the Clauser method, while the alternative fit yields a curve which passes directly through the lowest Reynolds number velocity gradient estimates of Purtell et al. The obvious conclusion is: extrapolations of fits to outside the range of the actual data should not be used since almost any desired result can be obtained.

It should be noted that the Clauser method results are also in close agreement with the theory, but only over a limited range in Reynolds number. This is not surprising, since as pointed out in section 29, the log profile (with its eddy viscosity origins) is, in fact, a first approximation to the mesolayer/overlap result around $y^+ = 100$ which is the point most often used in the determination. (In fact, the "effective" value of the von Karman constant from the parameters obtained above is almost exactly 0.42 at $R_\theta = 10^4$.) Unfortunately for very large and very small Reynolds number, the variation of C_i and γ with Reynolds number seriously erodes the accuracy of the approximation, so the friction estimates from this method are not reliable.

23 Empirical Velocity Profiles for the Wake and Buffer Regions.

A velocity profile valid over the entire flow can be obtained using equation 122 if empirical relations are introduced to account for the variation of $f_o(\overline{y}, \delta^+)$ outside the overlap layer and $f_i(y^+, \delta^+)$ inside it. This is exactly analogous to the use of the van Driest equation for the mixing length, the arctan profile for the Law of the Wall, or Coles Wake Function for the outer layer. All seek to use an empirical relation to splice together the various regions of the flow so that a continuous profile is obtained.

The part of the inner layer between the linear region and the inertial sublayer (the old log layer) has often been referred to as the buffer layer. The same terminology will be used here to refer to the region of adjustment from linear to the overlap region. A useful expression which makes this transition smoothly and in good agreement with the Purtell et al. data is given by

$$\frac{U}{u_*} = f_i(y^+) = y^+ exp(-d{y^+}^{5-\gamma}) + C_i {y^+}^{\gamma}(1 + \gamma a^+ {y^+}^{-1})[1 - exp(-d{y^+}^{5-\gamma})] \qquad (157)$$

The $y^{+5-\gamma}$-dependence of the exponentials allows not only the no-slip condition to be satisfied at the wall, but also the boundary conditions on the first four velocity derivatives if the damping parameter d is chosen as

$$d = \frac{c_4}{\gamma C_i a^+} \approx \frac{c_4}{C_{mi}} \tag{158}$$

where c_4 is the coefficient of the fourth order term in an expansion of the velocity at the wall (i.e., $u^+ = y^+ + c_4 {y^+}^4 + \cdots$). For the curves shown in this paper $d = 0.00002$ and the mesolayer constant was $C_{mi} \approx a^+ \approx -37$, corresponding to $c_4 = -0.00074$. The value for c_4 is very difficult to determine with any accuracy, but was estimated from the data of Purtell et al.

For that portion of the boundary layer outside the log layer, Coles 1956 defined a"wake function" to account for the difference between the actual velocity profile and the log behavior. Similarly, a wake function can be defined here for the outer flow by subtracting the overlap solution in outer variables from the velocity normalized by U_∞. The resulting wake function is given by

$$w(\overline{y}, \delta^+) = \frac{U}{U_\infty} - C_o \overline{y}^\gamma \tag{159}$$

(Note that the term "wake" is probably not a great choice given our new understanding of this region, since θ is not constant, but continues to increase with x.)

The topmost plot of Figure 11 shows the velocity profile data of Purtell et al. and Smith/Walker plotted as $U/U_\infty - C_o \overline{y}^\gamma$ versus $\overline{y}$. The overlap region (or power law layer) manifests itself as the region for which the velocity difference, $U/U_\infty - C_o \overline{y}^\gamma$, is zero. The sharp drop for small $\overline{y}$ occurs in the viscous sublayer, while the region for $\overline{y} > 0.1$ approximately is the 'wake'. Two features are evident: First, clearly there is an asymptotic wake function for $R_\theta > 3000$. Second, as the Reynolds number is reduced below this value, the wake begins to disappear, and is completely gone by $R_\theta = 500$ or so. This second phenomenon was noted in Figures 8, and has also been observed by others using the "log" formulation (eg. Murlis and Bradshaw 1982). It is due to the effects of viscosity on the outer flow discussed in section 11, and particularly the failure of the dissipation to achieve a q^3/L form at these Reynolds numbers.

The bottommost plot of Figure 11 shows that simply dividing the data plotted above by the factor $1 - C_o$ collapses all the profiles for all Reynolds numbers between $Re = 465$ and $Re = 48,292$ onto a single curve, at least for all values $\overline{y} \leq 1$ (to within the scatter of the data). The large scatter for the lowest Reynolds number data is because both the wake and the factor $1 - C_o$ are very close to zero. There is no theoretical justification for this factor, although its use can be motivated by noting that it provides exactly the amount necessary to adjust the contribution of the power law to unity at $\overline{y} = 1$. The previously noted increase toward unity of C_o for small values of R_θ insures that the wake vanishes in this limit.

The success of the simple scaling factor above in collapsing the wake to a single curve means that a single empirical expression to describe this region is possible. Ideally a closed form solution to the outer similarity equation for the mean flow would be available, say one based on an eddy viscosity, for example. In the absence of that, a simple (and easily integrable) expression for the wake function which accounts for the observed behavior is given by

$$w(\overline{y}) = (1 - C_o)\overline{y} \sin B\overline{y} \tag{160}$$

The parameter B and the upper limit of applicability of this relation, say y_m, can even be chosen to insure that $U/U_\infty = 1$ and $dU/dy = 0$ at $\overline{y} = y_m$. The disadvantage of this choice is that B depends on Reynolds number. An excellent compromise choice for all Reynolds numbers and $\overline{y} \leq 1$ is $B = 2.03$. The wake profile of equation 160 with this value for B is plotted together with the data in the bottom figure of Figure 11.

Substitution of equation 160 yields the complete outer velocity profile as

$$\frac{U}{U_\infty} = C_o \overline{y}^\gamma + (1 - C_o)\overline{y} \sin B\overline{y} \tag{161}$$

When combined with equations 122, 157 and the results for parameters determined earlier, an analytical function for the entire velocity profile can be obtained. The result is (in outer variables)

$$\frac{U}{U_\infty} = (1 - C_o)\overline{y}\sin B\overline{y} + \frac{u_*}{U_\infty}\{\overline{y}\delta^+ exp[-d(\overline{y}\delta^+)^{5-\gamma}] + C_i(\overline{y}\delta^+)^\gamma[1 + \gamma a^+(\overline{y}\delta^+)^{-1}][1 - exp\{-d(\overline{y}\delta^+)^{5-\gamma}\}]\} \tag{162}$$

Thus the entire velocity profile is described from $0 \leq \overline{y} \leq 1$, or for all values of y out to $y = \delta_{99}$, beyond which the wake function is no longer valid.

Figures 12 through 15 show the velocity data of Purtell et al. and Smith/Walker in outer variables, and the composite velocity of equation 162 using parameter values obtained in the preceding section. Note that the roll-off for small values of $\overline{y}$ in the Smith/Walker data (Figures 14 and 15) is due to their inability to measure close to the wall, which for some data sets is outside where the buffer region and mesolayer are located, hence the deviation from the composite solution for the first data point. Also, there is a slight underestimate (about 5%) around $y^+ = 10$ which probably represents a shortcoming of the exponential interpolation formula from the viscous sublayer to the the mesolayer.

Overall, the agreement of the composite profile must be considered quite good over the entire Reynolds number range, and throughout the entire boundary layer. And this agreement was assumed in the final regression which determined the parameters. It is rather remarkable that a theory with only four parameters and a single empirical function (h) can describe the entire boundary layer, including the shear stress, over more than two decades in Reynolds number. Recall that the old theory uses only three parameters, BUT it did not include the mesolayer parameter, AND most importantly, it treated both the shear stress and boundary layer thickness as variables to be determined for the best fit to each profile. No such juggling is necessary here, and the results would seem to provide a strong indication that both the theory and empirical interpolations are consistent with the data.

24 δ_* and θ

Figure 16 shows δ_*/δ, θ/δ, and the shape factor δ_*/θ versus both δ^+ (leftmost) and R_θ (rightmost). The data are from Smith/Walker (1959), Purtell et al. (1981), and Wieghardt (1943). The solid line indicates the theoretical values obtained by integrating numerically the theoretical profiles shown in Figures 12 - 15. As before, δ is taken to be δ_{99}. Clearly the data and theory are in good agreement. This is not surprising in view of the excellent agreement between the theoretical and experimental velocity profiles. The consistency of the results should restore some credibility to the much-maligned δ_{99} as a reasonable length scale. Note, however, the failure of the data from different experiments to perfectly overlap, indicating perhaps some residual effect of initial conditions which is also present in the c_f data.

It is convenient to have analytical formulae for the integral thicknesses, and this is possible. The empirical profiles given by equations 157 and 161 can be substituted directly into the integrals arising from the displacement and momentum integral thicknesses, equations 127, 128, 135, 136, and 137. The major contribution except for the lowest Reynolds number flows is due to the leading terms so that

$$\frac{\delta_*}{\delta} \approx -I_1 \tag{163}$$

$$\frac{\theta}{\delta} \approx -I_1 - I_3 \tag{164}$$

I_1 and I_3 can easily be integrated analytically to obtain

$$-I_1 = 1 - \frac{C_o}{1+\gamma} - (1 - C_o)I_6 \tag{165}$$

$$I_3 \approx 1 - \frac{2C_o}{1+\gamma} + \frac{C_o^2}{1+2\gamma} + (1 - C_o)^2[I_7 - 2I_6] \tag{166}$$

where

$$I_6 = \frac{\sin B - B cos B}{B^2} \tag{167}$$

$$I_7 = \frac{1}{6} - \frac{\sin 2B}{4B} - \frac{\cos 2B}{4B^2} + \frac{\sin 2B}{8B^3} \tag{168}$$

Using $B = 2.03$ implies that $I_6 = 0.436$ and $I_7 = 0.290$. These values yield

$$-I_1 = 1 - \frac{C_o}{1+\gamma} - 0.436(1 - C_o) \tag{169}$$

$$I_3 \approx [1 - 0.582(1 - C_o)^2] - \frac{2C_o}{1+\gamma} + \frac{C_o^2}{1+2\gamma} \tag{170}$$

from which δ_*/δ and θ/δ can easily be computed for any value of δ^+. The results are in near perfect agreement with both the data and the theoretical curve shown in Figure 16 for $\delta^+ > 2000$ or $R_\theta > 3000$ approximately. Below these values, the neglected terms from the inner region cause both δ_* and θ to be underestimated.

The asymptotic values of I_1 and I_3 can readily be estimated using the asymptotic values for γ and C_o obtained above. The result is $I_{1\infty} = -0.0894$ and $I_{3\infty} = 0.0128$. It follows immediately that asymptotically

$$\frac{\delta_*}{\delta} \rightarrow 0.0894 \tag{171}$$

$$\frac{\theta}{\delta} \rightarrow 0.0767 \tag{172}$$

$$H \rightarrow 1.17 \quad . \tag{173}$$

The asymptotic values are well below those values of the data in Figure 16 indicating again that the asymptotic boundary layer is reached only at much higher Reynolds numbers than for which data is available. Interestingly, the asymptotic value of H is very close to those obtained by Kempf (1932) (see also Smith and Walker 1959) on a pontoon at Reynolds numbers more than an order of magnitude above that of the data utilized here.

One advantage of having analytical expressions for δ_*/δ and θ/δ (or I_1 and I_3) is that δ^+ can be computed for given values of R_{δ_*} or R_θ. This was done to produce Figure 17 using equations 151, 163, 164 and the final approximate forms for I_1 and I_3 derived above. Also shown for comparison are the results from the profile integration. As expected the agreement is excellent above $\delta^+ > 2000$ and $R_\theta = 3000$.

Finally, Figure 18 shows R_θ versus R_x from a numerical integration of equation 143 using equation 88 with the constants determined earlier. Also shown in the figure are the data of Smith and Walker. No attempt has been made to adjust for virtual origin in x by choosing a non-zero value for R_o. Obviously, the agreement is excellent over the the entire range of the data. Thus, unlike in other theories, momentum appears to be conserved.

25 The Turbulence Quantities

There have been numerous papers written on the failure of the classical scaling laws to collapse the moments of fluctuating quantities in the wall region (by which is usually meant the "log" region as well as the buffer layer between it and the linear layer). Among the most troublesome quantities are the variances of the streamwise fluctuating velocities and the Reynolds stress (e.g., Klewicki and Falco 1989, Bradshaw 1990, Spalart 1988). More recently problems with the behavior of two-point correlations in the wall region have been noted by Blackwelder 1993. Klewicki and Falco conclude:

- "In general, inner variable normalizations of statistical profiles derived from the u and v fluctuations and the uv shear product do not produce invariant curves in the inner and near-wall regions of the boundary layer over the given R_θ range ($1000 < R_\theta < 5000$)."
- "Both the peak value and the y^+ position of the peak value of the u'/u_*, v'/u_*, and $< uv > /u_*^2$ profiles increased with increasing R_θ."
- "The present measurements support the hypothesis that for y^+ less than about 50 and with inner variable normalization, the statistical characteristics of single point spanwise vorticity measurements are invariant over the given R_θ range."

All of these observations for the single-point statistics are consistent with the theory put forth here. The reason quite simply is that the mesolayer and overlap layers can not be considered to be Reynolds number independent in either inner or outer variables for any *finite* Reynolds number. As a consequence, any measurements outside the viscous sublayer — roughly the linear region — and inside the deficit region, should be expected to display Reynolds number dependencies, whatever the quantity measured. For some quantities, like the mean velocity and vorticity, the Reynolds number dependence of the overlap range is hardly perceptible; for others, like the Reynolds stresses it is obvious. Since the velocity scale ratio, u_*/U_∞, varies more rapidly as the Reynolds number is reduced, these effects will be more pronounced at lower Reynolds numbers.

The theory put forth here has suggested that similarity of the turbulence quantities in the outer layer occurs only at Reynolds numbers high enough for the dissipation and Reynolds stresses to be effectively inviscid. While it is difficult to quantify this, it is generally assumed in turbulence that this occurs only when the turbulence Reynolds number, ul/ν, is on the order of 10^4 or larger (cf. Batchelor 1953). Since $u \sim u_*$ and $l \sim \theta$, this requires values of R_θ of 10^5 or larger for these conditions to be met. This is well beyond the range of any experiments to-date. The theory does suggest, however, some alternatives which might work before this is reached. These different scalings for the turbulence quantities in the outer layer follow directly from the transformed equations and depend on both u_* and U_∞. The normal stresses, for example, scale as U_∞^2, while the Reynolds shear stress scales (to first order) with u_*^2. Note that nothing should be expected to collapse in the overlap region since it depends on both u_* and U_∞. However the fact that u_*/U_∞ varies more slowly with increasing R_θ might lead to the erroneous conclusion that an asymptote is being approached.

Figure 19 shows the rms streamwise fluctuating velocity measurements of Purtell et al. 1981 normalized by U_∞. It is clear that the collapse is remarkable for $\overline{y} > 0.5$ for all Reynolds numbers, and the region of collapse moves towards the wall with increasing Reynolds numbers (just as for the mean velocity profiles above). The same data normalized with u_* appears in the paper of Purtell et al 1981 and there is little evidence of even a trend toward collapse in the outer part of the flow. A similar failure of the u_* scaling for the normal Reynolds stresses in the outer part of the flow was noted by Smith 1994. Even more significantly, Smith shows that the turbulence production term, $< -uv > dU/dy$, collapses in the outer region about the same when normalized by either $u_*^2 U_\infty/\delta$ or U_∞^3/δ, consistent with the fact that the outer layer is governed by two velocity scales and similarity is possible only in the limit when their ratio is constant.

Balint et al. 1991 show several attempts to collapse measurements of the mean square vorticity components over the outer layer, none of which is very successful. Most interesting though is that when the outer rms vorticity components were scaled with u_*^2/ν the order of the curves was reversed from when they were scaled with U_∞/δ. Thus the proper scaling (if there is one at all) is some combination of these. If the assumption of locally homogeneous turbulence is made, then the mean square vorticity is given by (v. George and Hussein 1991),

$$< \omega_i \omega_i >= \epsilon/\nu \tag{174}$$

where ϵ is the rate of dissipation of turbulence energy. Thus the vorticity in the outer layer should scale with the dissipation. From equations 48, 51 and 52 it follows that the dissipation scales as

$$D_s \sim U_\infty^3 \frac{d\delta}{dx} \sim U_\infty u_*^2 \tag{175}$$

Thus the rms vorticity in the outer region should scale as

$$< \omega_i \omega_i >^{1/2} \sim (u_*^2 U_\infty / \nu\delta)^{1/2} \tag{176}$$

Figure 20 shows a plot of the vorticity measurements of Klewicki and Falco 1990 using the outer scaling of equation 176. The values of wall shear stress used were those provided in the paper, and were quite close to the values computed from the friction law determined herein. The lack of collapse of the lowest Reynolds number data might be associated with the disappearance of the wake function noted above. In particular, the Reynolds number dependent constant D_o in equation 118 may be sufficiently different from its asymptotic value at this low Reynolds number so that u_*^2 alone is not the right parameter. Alternatively, it may represent a problem with the flow or the measurements (background vorticity or noise) since a constant subtracted from this data set yields curves much like those for the velocity deficit (Figure 3). Of course, u_* itself may simply be in error; or the proposed theory incorrect.

The two point velocity correlation measurements in the wall region of Blackwelder (1993) provide further substantiation for the ideas presented here. Briefly, it is observed that for small values of separation in horizontal planes, the two point correlations appear to collapse in wall variables, while for large separations they do not. Moreover, at large separations the magnitudes of the correlations still show a strong Reynolds number dependence, even when re-scaled with the usual outer variables, u_* and δ. In particular, the amount of correlation at large separations increases with Reynolds number. This behavior can easily be understood in the context of the new theory presented here. The large separations are primarily the "foot-print" of the outer boundary layer motions and so should scale in outer variables. If both inner and outer velocity scales were the same, then no Reynolds number dependence of the amplitude would be possible and the shape of the correlation at large separations should be Reynolds number independent. If, on the other hand, the outer velocity scale for the energy is not u_*^2, but U_∞^2 as suggested earlier, then the shape of the velocity correlation functions at large separations should exhibit the Reynolds number dependence observed.

The pressure fluctuations on the wall and near it have presented somewhat of a dilemma for the conventional theories because they do not scale in inner parameters, u_* and ν, alone. From the perspective of the theory presented here, these represent interesting examples of single point statistics which should not be expected to scale in local variables. This is because the pressure fluctuations are not governed by the same equations as the quantities for which the scaling arguments were derived. In fact, they are governed by a Poisson equation, the solution of which involves both an integral over the entire flow, and one over the wall (Batchelor 1967). Both of these integrals involve two points, the one under consideration, and the integration variable. Thus, only a two-point similarity analysis will suffice, and the considerations above are immediately applicable with the result that both inner and outer scales govern these quantities. Bradshaw 1967 hypothesized the existence of "inactive turbulent motions" and the "splat effect", the former recognizing that large energetic scales from outside the wall layer contribute to pressure fluctuations, and the latter accounting for the effect of the wall through the kinematic boundary condition. The idea of inactive motions is especially interesting because it anticipates the fact that the kinetic energy and Reynolds stress scale differently in the outer flow than in the inner. All these ideas follow naturally from the two point similarity considerations and the AIP, so unlike the classical approach, no new hypotheses need to be invoked to explain them.

Finally, the two point statistics also provide evidence for the mesolayer arguments put forth earlier. The one-dimensional velocity spectral measurements of Folz et al. (1996) show clearly the emergence of the $k^{-5/3}$ range for values of y^+ greater than a few hundred. Similar observations at much lower Reynolds number have been made by a number of investigators (e.g., Perry et al. 1985, Smith 1994).

26 Summary and Conclusions

A new theory has been set forth for the zero pressure–gradient boundary layer using an Asymptotic Invariance Principle and Near–Asymptotics. The equations for both the inner and outer regions of the boundary layer become asymptotically independent of the Reynolds number in the limit of infinite Reynolds number. In this limit (and only in this limit), these inner and outer equations admit to similarity solutions. These similarity solutions are used to determine the scaling parameters for finite Reynolds number. The fact that similarity solutions are strictly valid only for infinite Reynolds number means that *no scaling 'law' can work perfectly at finite Reynolds numbers.* Moreover, only the proposed scaling can be Reynolds number invariant in the infinite Reynolds number limit.

The outer boundary layer is found to be governed by a different scaling law than commonly believed. In particular, the velocity deficit in the outer layer scales as $(U - U_\infty)/U_\infty$. (To satisfy Galilean invariance, U_∞ can be replaced by the difference between the free stream velocity and the velocity of the surface.) The Reynolds shear stress in the outer layer, on the other hand, scales with $U_\infty^2 d\delta/dx$ which to first order is u_*^2, so that the outer layer is governed by two velocity scales. The classical inner scaling laws using u_* and ν

were found to be consistent with the similarity analysis, except that their region of applicability is less than previously believed.

By using Near-Asymptotics to examine how the inner and outer scaled velocity profiles behave for finite Reynolds number, the velocity in the overlap layer was shown to exhibit power law behavior, i.e., $U/u_* = C_i y^{+\gamma}$ and $U/U_\infty = C_o \overline{y}^\gamma$. The parameters C_o, C_i and γ are Reynolds number dependent, and only asymptotically constant. Thus, unlike the earlier theories, the overlap region is not Reynolds number invariant in either inner or outer variables, except in the infinite Reynolds number limit. The friction law was also shown to be of power law form; in particular, $u_*/U_\infty = (C_o/C_i)\delta^{+^{-\gamma}}$.

New scaling laws for some of the turbulence moments were derived from similarity considerations of the turbulence Reynolds stress equations. For the turbulence quantities in the outer layer to be asymptotically independent of Reynolds number, it was shown to be necessary that the asymptotic growth rate of the boundary layer be constant. Consequences of this are that the friction coefficient must be asymptotically constant. It was not clear from the theory or the data whether zero was an acceptable value of this constant, or whether the power exponent was itself asymptotically zero. The requirement for a finite energy dissipation rate in the limit appears to resolve the question by requiring that the power exponent be asymptotically non-zero, so the friction coefficient must be asymptotically zero. Regardless, arguments were presented that these limits are reached well beyond the Reynolds number range of existing experiments, i.e., $R_\theta > 10^5$.

By considering the role of viscosity in the single point and two point Reynolds stress equations, it was argued that there exists a *Mesolayer* in the region approximately defined by $10 < y^+ < 300$. In this region the overlap solutions *alone* do not apply because the *local* turbulence Reynolds number is too low. A simple turbulence model, valid in the range $30 < y^+ < 0.1\delta^+$ approximately, was used to derive a correction to the overlap velocity profile given in inner variables by C_{mi}/y^+ where from the data, $C_{mi} \approx -37$. This term only modifies the power law, but never dominates it, so there is no separate y^{-1} layer. Even so, because of the mesolayer, the overlap region does not even begin to evolve as a distinct region until $R_\theta \sim 10^4$ since below this value there is no region satisfying $300 < y^+ < 0.1\delta^+$, a necessary condition. In view of the simplicity of the model, better approximations to the mesolayer contribution are probably possible.

The theory was shown to be in general agreement with the bulk of the experimental data. From a single empirical relationship, $h = A/(\ln \delta^+)^\alpha$, determined from the data, both the power, γ, and the ratio C_o/C_i could be calculated analytically from a constraint relationship between γ and C_o/C_i . The asymptotic value of $C_o = 0.897$ is achieved in the experiments at about $R_\theta \sim 3 \times 10^3$, so beyond this the variation of C_i is known to within the accuracy and extent of the data. Thus the variation of all the parameters with Reynolds number is known, and the shear stress as well. The asymptotic values of $\gamma_\infty = 0.0362$ and $C_{i\infty} = 55$ are achieved at Reynolds numbers well beyond the range of the data. It is likely that the precise values for these parameters will change as better experiments become available.

The introduction of empiricism for the Reynolds number dependence of the power γ (or h) *at the very end of the analysis* must be contrasted with the traditional log theory where it appears at the very first step with the *assumption* of a velocity deficit law on which all subsequent arguments depend. It seems likely that even h can be determined from symmetry considerations in the future. It is clear, however, that the proposed form for $h(\delta^+)$ can not be exactly right, since the constants A and α depend on the definition of δ. An intriguing and invariant possibility is the $h = A'/(\ln \delta^+/\delta_o^+)^{\alpha'}$ where δ_o^+ is determined in some way by the initial (or upstream) conditions. There is some evidence among the different experiments for such a weak dependence on upstream conditions (e.g., the imperfect overlap of the skin friction, the integral boundary layer parameters, and even the values of γ, C_o and C_i between the Smith/Walker and Purtell experiments), but it was not possible to quantify this in the present study.

The power law dependence of the matched region derived here was suggested by Barenblatt (1978, 1993) from very different considerations. Unlike Barenblatt's inferences, however, the theory here suggests that pipe and channel flows will not show this behavior, and in fact the new theory allows a clear distinction to be made between the pipe or channel flow, and boundary layers (see Appendix I). Since the former must satisfy the homogeneous integral momentum equation, the pressure gradient and wall shear stress are not independent, and thus only one can enter the scaling (contrary to the assumption of Tennekes 1968, see also Tennekes and Lumley 1972). As a consequence, u_* is the correct scaling velocity for the core region and Millikan's analysis of it is correct. Thus the pipe is governed by a log law (even though it does not have a constant stress layer), while the zero pressure-gradient boundary layer is governed by a power law (even though it does).

It appears that the streamwise homogeneity of the pipe or channel flows dictates log layers, while the inhomogeneity of the boundary layer dictates power laws. Moreover, the Reynolds number of the flow does not change with downstream distance for the pipe/channel, but it evolves continuously in the boundary layer. As a consequence, the overlap and mesolayer regions are slowly moving away from the wall in physical variables, and the distance from the wall alone cannot characterize variations within it as for the pipe/channel. Prandtl's hypothesis that y, the distance from the wall, is the only length scale in the matched layer is thus only an approximation for boundary layers, while it is an exact asymptotic limit for pipe and channel flows.

That the boundary layer is only weakly inhomogeneous accounts for the fact that the log results have been close enough to be rationalized as correct. It is clear from the above that the wall layers of boundary layers and pipe/channel flows must be fundamentally different, however close they might be in practice. Because of the long period of time (more than 60 years) the log law theory has been believed to apply to turbulent boundary layers, it is natural to expect some resistance to any new theory which challenges it, no matter how well reasoned or argued. Part of the reason for acceptance of the old theory is that it has been believed to have been more or less consistent with the experimental velocity data which seemed to exhibit a logarithmic region. (Other philosophical reasons have arisen to justify it, like the principle of Reynolds number invariance, but it is the data itself which has been at the root of the faith.) It is, therefore, incumbent on any new theory to not only be internally consistent, but to explain how so many could have been so wrong for so long in believing the old. The paper has attempted to do both. At the very least, it is hoped that a strong motivation has been provided for a new generation of experiments over the entire range of Reynolds numbers.

27 Acknowledgments

The work reported here is a culmination of work initiated in 1981 while WKG was at RISOE, the Danish National Laboratory and continued while he was a Fulbright Scholar at the Technical University of Denmark in 1988. The paper was brought to its present state of understanding while WKG was a visiting scholar at Chalmers University, the Swedish Royal Institute, and the University of Glasgow in 1996. WKG is especially grateful for the opportunities to have worked in these stimulating environments, and to the many people who make them so. This work formed a portion of the PhD dissertation of LC, gratefully acknowledges the support of SUNY/Buffalo through a UB Minority Fellowship.

The authors would like to acknowledge the help of P. Knecht and D.B. Bower who participated in the early phases of this work. There are many others who provided helpful comments, data, and willingly took part in a useful dialogue (often by e-mail). At the risk of leaving someone out we would like to acknowledge the following: A. Alving, R. Adrian, P.H. Alfredsson, R.E.A. Arndt, J. Balint, R. Blackwelder, J. Citriniti, D. Ewing, M. Gad-el-Hak, M.M. Gibson, W.F. King J.C. Klewicki, J. Haridonitis, L. Kristensen, N.O. Jensen, A.V. Johansson, M. Landahl, P.S. Larsen, E. Lindborg, C.S. Liu, L. Löfdahl, A.E. Perry, P. Purtell J.M. van Oulden, A. Smits, J. Sonnenmeier, P. Spalart, D.B. Taulbee, J.M. Wallace, and M. Wosnik. It should not be assumed that any of these were in agreement with the ideas presented here. The "eye-witness" account (following a seminar by WKG at the U. Minn. in 1983) by Prof. E. Silberman (Emeritus) of Millikan's original presentation in 1938 proved to be crucial in unraveling the history, and in bringing this theory to its present state.

28 Appendix I: Turbulent Pipe and Channel Flow

It has been commonly assumed that turbulent channel and pipe flows have certain features in common with boundary layer flow; in particular the Law of the Wall, which has generally been taken to include the overlap region. It is the purpose of this brief discussion to show that this is not true, at least as far as the overlap region is concerned, since unlike boundary flows, these confined flows are indeed characterized by a logarithmic matched layer. The streamwise momentum equation for a fully developed two-dimensional channel flow at high Reynolds number reduces to

$$0 = -\frac{1}{\rho}\frac{dP}{dx} + \frac{\partial}{\partial y}\left[- < uv > + \nu\frac{\partial U}{\partial y}\right] \tag{177}$$

Like the boundary layer, the viscous term is negligible everywhere but very near the wall, so that the core (or outer) flow in the limit of infinite Reynolds number is governed by

$$0 = -\frac{1}{\rho}\frac{dP}{dx} + \frac{\partial}{\partial y} - < uv > \tag{178}$$

while the inner layer is governed to first order by equation 3.

It is obvious that the inner profiles must scale with u_* and ν; hence, a law of the wall (but not necessarily the same one). Because there is no imposed condition on the velocity, except at the wall, an outer scaling velocity must be sought from the parameters in the outer equation itself. Since there are only two, $-(1/\rho)dP/dx$ and R the channel half-width, only a single velocity can be formed; namely,

$$U_o = \left(-\frac{R}{\rho}\frac{dP}{dx}\right)^{1/2} \tag{179}$$

where R is the channel height (or pipe radius).

Unlike the developing boundary layer, the channel flow is homogeneous in the streamwise direction, so there is an exact balance between the wall shear stress acting on the walls, and the net pressure force acting across the flow. This equilibrium requires that

$$u_*^2 = -\frac{R}{\rho}\frac{dP}{dx} \tag{180}$$

which is just the square of equation 179 above. Therefore, the outer scale velocity is also u_*, and the outer and inner velocity scales are the same.

Thus the appropriate outer and inner scaling laws for the velocity profile are

$$\frac{U - U_c}{u_*} = f_o(\overline{y}, R^+) \tag{181}$$

and

$$\frac{U}{u_*} = f_i(y^+, R^+) \tag{182}$$

where the outer velocity has been referenced to the velocity at the centerline, U_c, to avoid the necessity of accounting for the change over the inner layer. The only other difference from the boundary layer is that the outer length scale is some measure of the width of the channel, say the half-width. As a consequence, the appropriate Reynolds number is $R^+ = u_* R/\nu$.

By defining

$$g(R^+) = \frac{u_*}{U_c} \tag{183}$$

the matching condition on the velocity becomes

$$\frac{1}{g(R^+)} + f_o({R^+}^{n-1}\tilde{y}, R^+) = f_i({R^+}^{n}\tilde{y}, R^+) \tag{184}$$

where the symbols R^+, $\tilde{y}$, and n have the same meanings as for the boundary layer discussed in the main body of the paper. Equating the velocity derivatives implies that

$$\overline{y}\frac{\partial f_o}{\partial \overline{y}} = y^+ \frac{\partial f_i}{\partial y^+} \tag{185}$$

Taking the partial derivative of equation 184 with respect to R^+ for fixed $\tilde{y}$ leads after some manipulation to

$$\overline{y}\left(\frac{\partial f_o}{\partial \overline{y}}\right)_{R^+} = -\frac{R^+}{g^2}\frac{dg}{dR^+} - R^+ \left\{ \left(\frac{\partial f_i}{\partial R^+}\right)_{y^+} - \left(\frac{\partial f_o}{\partial R^+}\right)_{\overline{y}} \right\} \tag{186}$$

Since, as before both f_o and f_i must become asymptotically independent of R^+ (or else the scaling is wrong), to leading order,

$$\overline{y}\frac{\partial f_o}{\partial \overline{y}} = \frac{1}{\kappa} \tag{187}$$

and

$$y^+ \frac{\partial f_i}{\partial y^+} = \frac{1}{\kappa} \tag{188}$$

where

$$\frac{1}{\kappa(R^+)} \equiv -\frac{R^+}{g^2}\frac{dg}{dR^+} \tag{189}$$

It follows immediately that

$$f_o(\overline{y}, R^+) = \frac{1}{\kappa(R^+)} ln\overline{y} + B_o(R^+) \tag{190}$$

$$f_i(y^+, R^+) = \frac{1}{\kappa(R^+)} ln y^+ + B_i(R^+) \tag{191}$$

and

$$\frac{1}{g(R^+)} = \frac{1}{\kappa(R^+)} ln R^+ + [B_i(R^+) - B_o(R^+)] \tag{192}$$

These are exactly the Millikan results (equations 14 – 16), but are more general since the parameters κ, B_i, and B_o need only be asymptotically constant. As for the boundary layer, they are easily shown to be subject to a constraint given in this case by

$$ln R^+ \frac{d(1/\kappa)}{dR^+} = -\frac{d(B_i - B_o)}{dR^+} \tag{193}$$

Thus the inner profile is quite different from that of the zero pressure–gradient boundary layer. This should come as no great surprise, since with the exception of the linear sublayer where $u^+ = y^+$ for any flow, the two flows are driven by very different mechanisms — the boundary layer by momentum transfer from the external flow, the channel flow by a pressure gradient. Obviously even the relatively small effect of this gradient on the near wall flow of the latter is quite different from having no gradient at all. Perhaps more importantly, for fixed Reynolds number, the matched layer remains at the same distance from the wall regardless of streamwise position because the flow is homogeneous in that direction.

It is easy to show that solutions to equation 193 must be of the form

$$B_o - B_i = (\frac{1}{\kappa} - \frac{1}{\kappa_\infty}) \ln R^+ - H(R^+) \tag{194}$$

where $H(R^+)$ must satisfy

$$\frac{1}{\kappa} = \frac{1}{\kappa_\infty} + \frac{dH}{d \ln R^+} \tag{195}$$

It follows that

$$\frac{U_c}{u_*} = \frac{1}{\kappa_\infty} \ln R^+ + H \tag{196}$$

These ideas have recently been applied by George and Castillo (1996) to the Princeton superpipe experiment (Zagarola 1996). The empirical function was found to be $H - (B_{i\infty} - B_{o\infty}) = A/(\ln R^+)^\alpha$ where $\alpha = 0.44$, $A = -0.0668$, $B_{i\infty} = 6.5$, $B_{o\infty} = -1.95$, and $\kappa_\infty = 0.447$.

29 Appendix III. A Mesolayer Model

It was Long 1976 (see Long and Chen 1982) who first argued for the existence of a mesolayer — but on very different physical grounds. He did not consider the turbulence energy equation, but instead only the mean momentum equation. From it he argued that some residual viscous stress must be retained in addition to the Reynolds stress, and used this to define a meso-length scale which varied as the square root of the flow Reynolds number. All subsequent deductions were based on matching four flow regions, one of which was characterized by this new length scale. The suspicions that a new layer involving viscosity and inertia was needed between the overlap and viscous layers has proven to be quite insightful. The arguments, however, can not be justified since there is simply no physical basis for arguing that the viscous stress must be important *in the equations for the mean flow.* In fact it is negligible outside of $y^+ \approx 10$. It was argued in the preceding section that viscosity enters the dynamics of the mesolayer only through its effect on the energy cascade, and that is reflected in the nature of the dissipation, and in turn in the component Reynolds stress equations.

It is easy to show that no new length scale is necessary to account for this dissipation effect. The whole reason for the existence of this mesolayer is that the local turbulence Reynolds number near the wall can never be large enough for the dissipation to become inviscid. Near the bottom of the constant stress layer, the scales of the energy-containing eddies and those at which the energy is dissipated will be nearly the same size, and in this limit $\epsilon \sim \nu q^2/L^2$ where L is typically about equal to y, the distance from the wall. At the outer part of the constant stress layer, the required scale separation will have been achieved — if the flow Reynolds number is high enough — so the dissipation is nearly inviscid and thus $\epsilon \sim q^3/y$. The essence of the mesolayer is that neither of these limits applies and a transition from one to the other is occurring. Thus in the mesolayer, $\nu q^2/y^2 \sim q^3/y$, and it follows immediately the length scale for the mesolayer is just proportional to $y \sim \nu/q \approx \nu/u_*$. But this just says that the mesolayer length scale is proportional to the viscous one. It does show clearly, however, that the mesolayer is bounded by relatively fixed values of y^+ as argued earlier, the slight variation being due to the fact the ratio q/u_* has a weak Reynolds number dependence (for a given of y^+), and is constant only in the limit.

In the mesolayer, the nature of the dissipation is changing with distance from the wall as the local Reynolds number, y^+, increases. And it is this evolution from low to high Reynolds number dissipation which provides a clue for building a model for at least part of the mesolayer. Note that the analysis below is a *physical model* based on an assumed form of the dissipation, and is therefore quite distinct from the AIP and Near-Asymptotics approach described earlier. It will be shown, however, to be consistent with the latter, and to lend considerable insight into it.

Since it is the dissipation itself which creates the mesolayer, it is reasonable to begin by assuming a form for how the dissipation changes with Reynolds number, and then pursuing its logical consequences. A simple model incorporating both the high and low Reynolds number dissipation limits is

$$\epsilon = c_1 \frac{q^3}{L} + c_2 \nu \frac{q^2}{L^2} \tag{197}$$

For very high values of qL/ν the first term dominates, but the second overwhelms it when $qL/\nu << c_2/c_1$. Variations on this idea have appeared in numerous low Reynolds number turbulence models (cf. Hanjalic and Launder 1973, Reynolds 1976, Rodi 1993). Since only the near wall region is of interest, it is appropriate to take $L = y$, as done by many single equation turbulence modellers for the near wall region.

As long as $y^+ > 30$, the kinetic energy equation for the turbulence reduces to simply a balance between production and dissipation, the turbulence transport terms being negligible; i.e.,

$$- < uv > \frac{\partial U}{\partial y} = \epsilon \tag{198}$$

The turbulence transport terms are certainly not negligible in the region $10 < y^+ < 30$ which is also part of the mesolayer, so any success of the model in this region must be regarded as fortuitous. (The authors are grateful to Drs. M.M. Gibson and W.P. Jones of the Imperial College of London for helpful discussion about this region.)

Now consistent with the single equation turbulence model is the assumption that the Reynolds stress can be modelled with an eddy viscosity acting on the mean velocity gradient; i.e.,

$$-<uv>= \nu_t \frac{dU}{dy} \tag{199}$$

The usual choice of turbulence modellers is (Rodi 1993)

$$\nu_t = c_3^2 \frac{q^4}{\epsilon} \tag{200}$$

Substituting the dissipation and Reynolds stress models into the energy balance of equation 198, dividing by q^4/ϵ, and taking the square root yields

$$\frac{\partial U}{\partial y} = \left(\frac{c_1}{c_3}\right) \frac{q}{y} + \left(\frac{c_2}{c_3}\right) \frac{\nu}{y^2} \tag{201}$$

or in inner variables,

$$\frac{du^+}{dy^+} = \left(\frac{c_1}{c_3}\right) \left(\frac{q}{u_*}\right) {y^+}^{-1} + \left(\frac{c_2}{c_3}\right) {y^+}^{-2} \tag{202}$$

Obviously it is the factor q/u_* which determines whether the first term on the right hand side integrates to a logarithm or a power law (or something else).

It is easy to show that in the overlap region (just as for the Reynolds shear stress considered earlier), $q^2/u_*^2 = C_q(\delta^+){y^+}^{\alpha}$. [9] The overlap velocity power law can be recovered in the limit of large y^+ if $\alpha = 2\gamma$, which is, in fact, consistent with the eddy viscosity and dissipation modeling assumptions above. Substitution into equation 202 and integration yields immediately,

$$u^+ = C_i {y^+}^{\gamma} + C_{mi} {y^+}^{-1} \tag{203}$$

The integration constant has been taken as identically zero to correspond to the previously derived overlap layer as $y^+ \to \infty$, and the other parameters have been collected into $C_i(\delta^+)$ and C_{mi}. The second term is unaffected by the behavior of q/u_*; hence there is reason to hope that it may be the same for all wall-bounded flows. (Note that equation 203 can be derived using only the overlap characteristics without reference to an eddy viscosity model.)

Thus the additional contribution of the mesolayer to the velocity profile (in inner variables) is $C_{mi}{y^+}^{-1}$. The parameter C_{mi} must be negative and should be nearly constant. It will be seen later in Part II that that because of the relative values of C_i and C_{mi} there is no region where the second term dominates, at least where the assumptions are valid. Therefore there will be no ${y^+}^{-1}$-layer, only a modified power law region. Moreover, because of this, the first term in equation 203 will be clearly visible only when the second is negligible. Since this is not the case for many of the low and moderate Reynolds number experiments, it will not be possible to even identify the parameters C_i, C_o, and γ for most of the data *without first accounting for the mesolayer contribution.* (Note that similar considerations for a channel or pipe flow yield a similar term added to the log profile.)

[9] This is a consequence of the fact that the inner and outer scales are different. Pipe and channel flows, however, show a logarithmic dependence for all quantities.

********* By comparing the second terms of equations 104 and 105, and ignoring the slight difference in exponent (since $\gamma << 1$), an approximate relation between a^+ and C_{mi} can be obtained as

$$C_{mi} \approx \gamma C_i a^+ \tag{204}$$

or

$$a^+ \approx \frac{C_{mi}}{\gamma C_i} \tag{205}$$

For the range of Reynolds number of the KEP data (and the boundary layer data as well), $a^+ \approx C_{mi}$ since $\gamma C_i \approx 1$. The mesolayer parameter, C_{mi} was found by GCK to be constant at approximately -37 which corresponds to $a^+ \approx -37$ also. ************

Equation 203 can be expressed in outer variables as

$$\frac{U}{U_\infty} = C_o \overline{y}^\gamma + C_{mo} \overline{y}^{-1} \tag{206}$$

where

$$C_{mo} = C_{mi} \delta^{+^{-1}} \frac{u_*}{U_m} = C_{mi} \frac{C_o}{C_i} \delta^{+^{-(1+\gamma)}} \tag{207}$$

Obviously if C_{mi} is constant, C_{mo} is not.

Since the constant stress layer for much of the data under consideration extends to only values of y^+ of a few hundred, inclusion of the mesolayer profile into the data analysis significantly modifies the conclusions about where the power law overlap region is located, as well as the values deduced for the parameters C_o, C_i, and γ. Both the inner and outer expressions will be utilized in the Part II to analyze the velocity profile data.

Before leaving this section it is interesting to note that equation 202 offers another insight as why the familiar log profile has survived so long. Suppose there were a region in the boundary layer for which production were not only approximately equal to the dissipation, but for which the ratio q/u_* were also approximately constant. If this region were at sufficiently large values of y^+ for the second term to be relatively small, then to a first approximation $du^+/dy^+ \sim 1/y^+$. Thus the profile corresponds exactly to that originally deduced by Prandtl (1932) from an viscosity hypothesis. These assumptions above are satisfied only over a narrow region for boundary layer flows ($50 < y^+ < 150$), but this is exactly the region where the log law is known to work best in boundary layer flows (Bradshaw and Huang 1995). These same authors note the seemingly paradoxical facts that the log profile is remarkably 'resilient', but its range of validity does not seem to increase with increasing Reynolds number like a proper overlap solution (or like pipe or channel flow). In fact, the overlap plus mesolayer profile derived above and the old log law are nearly indistinguishable over this limited range. All of these observations are consistent with the interpretation that the boundary layer 'log region' is in fact just a portion of the mesolayer. Thus Prandtl's log law is preserved, but only as an approximation over a limited range. Clauser's identification of this region with Millikan's matched layer is, however, clearly incorrect.

30 Appendix IV: Optimization Method of Data Analysis

In order to use all of the available information to determine C_o, C_i, γ, and C_{mi} the following formal optimization was utilized. The optimization is based on the composite profile of equation 162. This was used to calculate the integral thicknesses from equations 125 and 138. It was assumed that the shear stress measurements were unreliable, and so it was calculated using the theoretical relation, equation 84, as part of the optimization (except as noted below).

For the initial optimizations, one with the mesolayer term and the other without, the objective function to be minimized *for each profile* was

$$obj = ABS(\frac{U}{U_{comp}} - 1) \tag{208}$$

subject to the following equality constraints

$$H1 = \frac{\delta_*}{\delta_{*theor}} = 1 \tag{209}$$

$$H2 = \frac{\delta}{\delta_{theor}} = 1 \tag{210}$$

$$H3 = \frac{H}{H_{theor}} = 1 \tag{211}$$

Side constraints imposed on the design variables were:

$$C_o > 0 \tag{212}$$

$$C_i > 0 \tag{213}$$

$$\gamma > 0 \tag{214}$$

This method had the considerable advantage of not depending on any c_f measurements or linear layer velocity measurements. However, for the Smith/Walker data there were not sufficient points near the wall to determine the shear stress in this manner, Therefore for this data (only) an additional constraint was added; i.e.,

$$H4 = \frac{c_f}{c_{ftheor}} = 1 \tag{215}$$

where the values of c_f used were determined from the momentum integral as described in the text. The result of this data was used to determine the form of the h-function as $h = A/(\ln \delta^+)^\alpha$

The final optimization eliminated the integral and shear stress constraints and used instead the empirical h-function (with undetermined coefficients) and the resulting exact solutions for γ and C_o/C_i. The values of the parameters A, α, $C_{o\infty}$, $C_{i\infty}$, and γ_∞ were optimized by considering all of the profile data simultaneously, hence the smooth curve in Figure 7.

References

[1] Abrahamsson, H. private communication (1996).

[2] Abramowitz, M. and I.A. Stegun. *Handbook of Mathematical Functions*, Dover NY (1965).

[3] Balint, J-L, Wallace, J.M. and Vukoslavcevic, P. The velocity and vorticity vector fields of a turbulent boundary layer. Part 2. Statistical properties, *J. Fluid Mech., 228*, 53 – 86 (1991).

[4] Barenblatt, G.J. *Similarity, Self-similarity, and Intermediate Asymptotics*, Plenum, NY (1978).

[5] Barenblatt, G.J. Scaling laws for fully developed shear flow. Part I, Basic hypotheses and analysis. *J. Fluid Mech.*, vol. 248, 513 –520 (1993).

[6] Barenblatt, G.J. Scaling Laws for Fully Developed Flow in Pipes, to appear in *Applied Mechanics Reviews* (1997).

[7] Blackwelder, R Presentation at the Symposium on Turbulence, Asilomar Conference Center, Monterey, CA (1993).

[8] Blackwelder, R.F. and J.H. Haridonitis Scaling of the bursting frequency in turbulent boundary layers, *J. Fluid Mech.*, 132, 87 – 103 (1983).

[9] Bradshaw, P. Turbulence: The Chief Outstanding Difficulty of Our Subject, 3rd Stewartson Mem. Lecture, Symp. on Numerical and Physical Aspects of Aerodynamic Flows, Long Beach, CA (1992).

[10] Bradshaw, P. and Huang, G.P. The Law of the Wall in Turbulent Flows, *Osborne Reynolds Anniversary Symposium*, Univ. Manchester (1995).

[11] Bradshaw, P., Launder, B. and Lumley, J.L. CTTM, AIAA 91 - 0215 (1991).

[12] Bush, W.B. and Fendell, F.E. Asymptotic Analysis of Turbulent Channel and Boundary Layer Flow, *J. Fluid Mech.*, 56, 657 –681 (1974).

[13] Clauser, F.H. The turbulent boundary layer, *Adv. Appl. Mech.*, IV, 1-51 (1954).

[14] Cole, J.D. and Kevorkian *Perturbation Methods in Applied Mathematics*, Springer, NY (1981).

[15] Coles, D.C. The Turbulent Boundary Layer in a Compressible Fluid, Rand Report R-403-PR (1962).

[16] Coles, D.C. The Law of the Wake in Turbulent Boundary Layers, *J. Fluid Mech.*, 1, 191 – 226 (1956).

[17] Coles, D.C. in Kline, et al., 1968 AFOSR-IFP-Stanford Conf. Computation Turbulent Boundary Layers, Stanford Univ. Dept. Mech. Engr., Palo Alto, CA (1968).

[18] Fernholz, H.H., Krause, E., Nockermann, M., and Schober, M. Comparative measurements in the canonical boundary layer at $Re_{\delta_2} < 6 \times 10^4$ on the wall of the German-Dutch windtunnel. *Phys. Flds., 7*, 1275 –1281 (1995).

[19] Folz, A., Ong, L. and Wallace, J.M. Reynolds number Effects and Dissipation Physics in the Atmospheric Boundary Layer, presentation at APS/DFD, Syracuse, NY (1996).

[20] Frisch, U. *Turbulence: The Legacy of A.N. Kolmogorov*, Cambridge Univ. Press (1995).

[21] Gad-el-Hak, M. and Bandyopadhyay, P.R. Reynolds Number Effects in Wall-Bounded Flows, *Applied Mech. Rev., 47*, 307 –3365 (1994).

[22] George, W.K. Self-Preservation of Turbulent Flows and Its Relation to Initial Conditions and Coherent Structures, in *Advances in Turbulence* (George and Arndt, eds.), 39 – 73, Hemisphere, NY (1989).

[23] George, W.K. Some New Ideas for Similarity of Turbulent Shear Flows, *Proc. 1994 Conf. on Turbulence and Turbulent Heat Transfer*, Hanjalic and Peira, eds, Elsevier, Amsterdam (1995).

[24] George, W.K. and Capp, S.C. A Theory for Natural Convection Boundary Layers Next to Heated Vertical Surfaces, *Int. J. Heat and Mass Transfer, 22*, 813-826 (1979).

[25] George, W.K., Knecht, P. and Castillo, L. The Zero-Pressure Gradient Boundary Layer Revisited, *13th Symposium on Turbulence* (Reed, X.B., ed.), Rolla, Mo. (1992).

[26] George, W.K. and Castillo, L. Boundary Layers with Pressure Gradient: Another Look at the Equilibrium Boundary Layer, *Near Wall Turbulent Flows*, (So, R.M.C. et al. editors), 901 – 910, Elsevier, NY (1993).

[27] George, W.K. and Castillo, L. A Theory for Turbulent Pipe and Channel Flows. Prepared for *Disquitiones Mechanicae*, Univ. of Ill., Oct 1996 (to be distributed as a TAM report from Dept of Theor. and Appl. Mech., U. of Ill.), (1996).

[28] George, W.K. and Hussein, H.J. Locally Axisymmetric Turbulence, *J. Fluid Mech.*, 233, 1 – 23 (1991).

[29] Hama, F.R. Boundary-Layer Characteristics for Smooth and Rough Surfaces, *Trans. Soc. Nav. Archt. Marine Engrs.*, 62, 333 – 358 (1954).

[30] Hanjalic, K. and B.E. Launder Contribution towards a Reynolds stress closure for low-Reynolds number turbulence, J. Fluid Mech., 74, 34 – 39 (1974).

[31] Hinze, J.O. *Turbulence*, McGraw Hill, NY (1975).

[32] von Karman, T. Mechanische Ähnlichkeit und Turbulenz, *Nachr. Ges. Wiss.*, Göttingen, pp 68 (1930).

[33] Karlsson, R. Eriksson, J, and Persson, J. An Experimental Study of a two-dimensional Plane Turbulent Wall Jet. Rept VU-S 93:B36, Vattenfall Utveckling, Alvekarleby, Sweden (1993).

[34] von Kempf, G. Weitere Reibungsergebnisse an ebenen glatten und rauhen Flächen. Hydrodynamische Probleme des Schiffsantriebs, vol. 1, 74-82 (1932).

[35] King, W.F. private communication (1993)

[36] Klewicki, J.C. Ph.D. dissertation, Mich. State U. (1989).

[37] Klewicki, J.C. and Falco, R.E. On accurately measuring statistics associated with small-scale structure in turbulent boundary layers using hot-wire probes. *J. Fluid Mech., 219*, 119 – 142 (1990).

[38] Klewicki, J.C. and Falco, R.E. Effects of Reynolds Number on Turbulent Boundary Layer Statistics, Rep. TSL-89-04 Michigan State University, E. Lansing MI (1989).

[39] Kline, S.J., Reynolds, W.C., Schraub, F.A. and Rundstatler, P.W. The Structure of Turbulent Boundary Layers *J. Fluid Mech.*, 30, 4, 741 – 773 (1967).

[40] Knecht, P. An Investigation of the Matched Layer in a Turbulent Boundary Layer with Zero Pressure Gradient, MS Thesis, Dept. Mech. & Aersp. Engr., SUNY/Buffalo, Buffalo, NY (1990).

[41] Long, R.R. presentation at Naval Hydrodynamics Mtg., Wash., DC., June 1976.

[42] Long, R.R. and Chen, T.-C. Experimental Evidence for the Existence of the 'Mesolayer' in Turbulent Systems, *J. Fluid Mech.*, 105, 19 – 59 (1981).

[43] Monin, A.S. and Yaglom, A.M. *Statistical Fluid Mechanics*, MIT Press, Cambridge, MA (1971).

[44] Millikan, C.M. A critical discussion of turbulent flows in channels and circular tubes, *Proc. 5th Int. Congr. Appl. Mech.*, 386 – 392, Wiley, NY (1938).

[45] Murlis, J.T. and Bradshaw, P. *J. Fluid Mech., 122*, 13 (1982).

[46] Nagib, H. and Hites, M. High Reynolds Number Boundary-Layer Measurements in the NDF, AIAA 95-0786, 33rd Aero. Sci. Mtg., Reno, (1995).

[47] Nikuradze, J. Gesetzmaessigkeiten der turbulenten Stroemung in glatten Rohren, *VDI Forschungheft, 356* (1932).

[48] Oberlack, M. Lie group analysis of turbulent shear flow. (to be submitted for publication), (1996).

[49] Panton, R. Scaling Turbulent Wall Layers *J. Fluids Engr., 112*, 425 – 432.

[50] Perry, A.E. and Abell, C.J. *J. Fluid Mech., 67*, 257 (1975).

[51] Perry, A.E., Lim, K.L. and Henbest, S.M. A Spectral Analysis fo Smooth Flat-Plate Boundary Layers. In *Proceedings of the Fifth Symposium on Turbulent Shear Flows*, Cornell University (1985).

[52] Prandtl, L. Zur Turbulenten Strömung in Rohren und längs Platten, *Ergeb. Aerod. Versuch Göttingen*, IV Lieferung, 18 (1932).

[53] Purtell, L.P., Klebanoff, P.S. and Buckley, F.T. Turbulent Boundary Layer at Low Reynolds Number, *Phys. Flds.*, 24, 802 – 811 (1981).

[54] Reynolds, W.C. Computation of Turbulent Flows, Ann. Rev. of Fluid Mech., 8, 183 –208 (1976)

[55] Schlicting, H. *Boundary Layer Theory*, McGraw-Hill, NY (1968).

[56] Schultz-Grunow, F. New Frictional Resistance Law for Smooth Plates, NACA Tech. Mem. 986 (transl. from Neues Reibungswiderstandsgesetz fuer glatte Platten, *Luftfahrtforsch.*, 17, 8, 239 – 246, 1940), (1941).

[57] Smith, R.W. Effect of Reynolds Number on the Stucture of Turbulent Boundary Layers, Ph.D. Diss., Princeton U. (1994).

[58] Smith, D.W. and Walker, J.H. Skin-friction measurements in incompressible flow, NACA Rept. R 26 (1959).

[59] A.J. Smits and J.P. Dussauge, Turbulent shear layers in supersonic flows, AIP Press, ISBN 1-56396-260-8 (1996).

[60] Spalart, P.R. Direct simulation of a turbulent boundary layer up to $R_\theta = 1410$, *J. Fluid Mech., 187*, 61 – 98 (1988).

[61] Sreenivasan, K.R. The turbulent boundary layer, In *Frontiers in Experimental Fluid Mechanics*, (ed. Gad-el-Hak), 159 – 209 (1989).

[62] Stanton, T.E. and Pannell, J.R. Similarity of motion in relation to the surface friction of fluids, *Phil. Trans. Roy. Soc. A*, 214, p 199.

[63] Tennekes, H. and Lumley, J.L. *A First Course in Turbulence*, MIT, Cambridge (1972).

[64] Tennekes, H. Outline of a Second-Order Theory of Turbulent Pipe Flow,*AIAA Jour.*, 6, 1735 – 1740 (1968).

[65] Van Dyke, M. *Perturbation Methods in Fluid Mechanics*, Parabolic Press, Stanford, Ca. (1964).

[66] Zagarola, M.V. Ph.D. dissertation, Princeton Univ. (1996).