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## POLYNOMIAL CALIBRATIONS FOR HOT WIRES IN THERMALLY-VARYING FLOWS


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### Abstract

An alternative to the traditional forms for hot-wires calibrations is presented which expresses the velocity as a function of voltage in the form of a polynomial

$$U = \sum_{n=o}^{N} A_n E^n \quad \text{or} \quad Re = \sum_{n=o}^{N} A_n' N_u^n$$

where the coefficients are velocity independent. These forms have the advantage that the velocity can be calculated directly (and recursively) from the measured voltages once the coefficients are determined—a significant advantage for the computer implementation. Moreover, the coefficients occur linearly and can be determined by linear regression. Since the primary errors in calibration are usually in the determination of velocity whereas the voltage can be more accurately determined, the regression is properly directed to minimize the effect of measurement error—unlike the King's Law expressions and polynomials expressing voltage as function of velocity. Finally the quasi-linearization of the above expressions is discussed.


### Nomenclature

| | |
|---|---|
| $A$ | empirical coefficient |
| $A_n$ | empirical coefficient in eqn. 7,8 |
| $B$ | empirical coefficient |
| $C_n$ | empirical coefficients in eqn. 10 |
| $d$ | wire diameter |
| $E$ | CTA output voltage |
| $E_o$ | CTA output voltage at hypothetical zero velocity |
| $h$ | heat transfer coefficient defined by eqn. 4 |
| $k$ | thermal conductivity of fluid |
| $K$ | Knudsen number, $=d/\lambda$ |
| $m$ | exponent in temperature loading factor (eqn. 5,14) |
| $Nu$ | Nusselt number defined by eqn. 2 |
| $n$ | empirical exponent |
| $P$ | empirically determined exponent in eqn. 5 |
| $q$ | empirically determined exponent in eqn. 5 |
| $q_w$ | heat transfer rate per unit area from wire |
| $Re$ | Reynolds number defined by eqn. 2 |
| $t$ | fluctuating ambient temperature |
| $T_f$ | film temperature, $=(T_w+T_a)/2$ |
| $T, T_a$ | temperature of ambient fluid |
| $\nu$ | kinematic viscosity |
| $\lambda$ | mean free path |

### Introduction

The purpose of this article is to discuss the use of a polynomial for hot wire velocity calibration and to present some of its advantages over the more commonly used King's Law type expressions. While some of these advantages are well-known (for example, ease of computation on a computer), others are not so well-known. Moreover, there is a natural hesitancy among many in the flow measuring community to deviate from King's Law linearization because of its long-standing acceptance. While this may be reasonable when using analog linearizers which one already has at his disposal, it is suggested here that there are definite advantages to be gained by using a polynomial scheme—both for ease of implementation and accuracy. This is especially true when implementing a digital linearization or using a quasi-linearization technique.

### King's Law

In 1906 King obtained the following analytical solution to the problem of heat transfer from potential flow around a cylinder:

$$Nu = A + BRe^{1/2} \tag{1}$$

where

$$Nu = \frac{hd}{k} \text{ Nusselt Number} \tag{2}$$

$$Re = \frac{Ud}{\nu} \text{ Reynolds Number} \tag{3}$$

$$h = \frac{q_w}{T_w - T_a} \text{ Heat Transfer Coefficient} \quad (4)$$

and where d is the wire diameter, k is the thermal conductivity of the fluid, $\nu$ is the kinematic viscosity of the fluid, $q_w$ is the heat transfer rate per unit area from the wire, $T_w$ and $T_a$ are the wire surface and ambient fluid temperatures respectively.

While potential flow around a cylinder has little to do with the flow around a typical hot wire ($10^{-2} < Re < 100$), the form of King's Law has been retained in many of the subsequent attempts to establish empirical laws. These are reviewed in detail in McAdams (1954) and Collis and Williams (1959). A commonly used expression is a composite of those due to Kramers (1946) and Collis and Williams (c.f. Hinze 1975); namely,

$$Nu \left(\frac{T_w}{T_f}\right)^m = A\ Pr^p + B\ Pr^q\ Re^n \quad (5)$$

where Pr is the fluid Prandtl number and $(T_m/T_f)^m$ is a loading factor which compensates for the variation in the thermal properties of the fluid in the thermal boundary layer of the wire. Typically m=0.17, p=0.2, q=0.33 and $0.45 \leq n \leq 0.5$, the latter depending on Reynolds number. Additional terms or alternate expressions are needed at very low Reynolds numbers where the character of the equations is dominated by viscosity. An example of the former is the linear term often added at low velocities; the latter is illustrated by the Oseen-type logarithmic calibration used by Collis and Williams (1959) and George et al. (1977).

In rarefied gases or with very fine wires (less than 1 μm dia. in air), the fact that the wire diameter and the mean free path $\lambda$ are of the same order of magnitude is responsible for a breakdown in the continuum approximation. This can give rise to a dependence of the calibration on Knudsen number $K=d/\lambda$ (see for example, Collis and Williams 1959). Additional problems can occur at very low velocities where natural convection from the wire can dominate convective effects, and effectively limit the lower range of applicability of the calibration. These have been discussed in some detail by Hollasch and Gebhart (1972) and Warpinski et al. (1972).

## The Advantages of a Polynomial

While the above expressions (and others similar to them) can be used in a variety of applications with success, they are always awkward and difficult to implement--both to obtain the calibration curve and to apply it. The principal reason for this is the manner in which the voltage depends on the velocity. This is easily seen by the constant ambient temperature version of equation (5) which reduces for a given wire to

$$E^2 = A + BU^n \quad (6)$$

where E is the bridge top voltage typically measured in constant temperature anemometer (CTA) applications.

Performing the calibration is complicated by the fact that the determination of A is not as simple as measuring E with U=0, because of the natural convection effects mentioned above. This uncertainty is carried through the analysis since B and n can be determined only after A is determined and subtracted. Moreover, it should be noted that the usual linear least-squares type analysis is not applicable to this expression--firstly because it is not a linear equation, and secondly, because the principal uncertainty in the calibration measurements is usually in U and not in E.

Implementation of even the simple form of King's Law given in equation (6) is not straightforward either. Calculating the velocities from the measured voltage requires a time-consuming process on a computer, and analog conversion requires sophisticated circuitry.

The idea and practice of using a polynomial in both digital and analog linearization is not new. Cheesewright (1972) discussed digitally linearizing hot wire signals on a large computer, and commercial polynomial linearizers have been available for a number of years.

In its simplest form, the velocity is expressed as the sum of powers of the voltages, that is

$$U = \sum_{n=o}^{N} A_n E^n \quad (7)$$

The principal advantages are two-fold: Firstly, a linear-least squares can be directly applied since the coefficients occur linearly and since the principal uncertainty in the observations is on the left, namely U. Secondly, application of the calibration to measured data is straightforward and involves only recursive multiplication of the measured voltages.

An alternative form which is equivalent to equation (7) but is more convenient for analog implementation uses the offset voltage $E-E_{ref}$ instead; thus

$$U = \sum_{n=o}^{N} A_n (E-E_{ref})^n \quad (8)$$

where $E_{ref}$ can be chosen for convenience. A good choice for $E_{ref}$ is either the 'zero' velocity voltage or a midrange voltage. (Note that for reasons which will become clear in the next section it is sometimes preferable to work with equation (7) even if an offset is used prior to digitization. This is easily accomplished by adding the offset to the numbers stored in the computer.)

Because of the convenience of using a linear least-squares or other statistical algorithm for determining the coefficients (they can be carried out even on a hand calculator), it is no longer necessary to try to infer or measure the anemometer output at zero velocity. Thus the calibration can be entirely performed in the region where the wire will be used, and the natural convection regime avoided entirely. (In fact, this is the manner in which any calibration should be used, regardless of curve fit used.)

It is, of course, no surprise that polynomials can also provide superior fits to the calibration data, since by increasing the order of the polynomial the number of adjustable coefficients is also increased. However, experience in our laboratory indicates that there is little to be gained by going beyond the fourth order, i.e.,

$$U = A_o + A_1E + A_2E^2 + E_3E^3 + A_4E^4 \quad (9)$$

Independent studies at the Illinois Institute of Technology by Wlezien (1979) have also shown this choice to be superior over a wide range of velocities to other calibrations commonly in use.

## A Polynomial Heat Transfer Law

Equations (7)-(9) have the principal disadvantage that the coefficients in them are temperature dependent. This can be contrasted with equations (1) and (5) which are presumed to be valid over a wide range of temperatures. There is, of course, no reason why a 'heat transfer law' cannot be postulated which is based on a power law. Therefore we propose that the Reynolds number be expressed in half-powers of the Nusselt number; that is,

$$Re = \sum_{n=o}^{N} C_n Nu^{n/2} \quad (10)$$

The coefficients $C_n$ are now temperature independent and the temperature dependence enters entirely through the variations of $\nu$ and $k$ in the Nusselt and Reynolds numbers, and through the direct dependence of Nu on $T_w - T_a$.

We have had great success in our laboratory with the fourth order polynomials in $Nu^{1/2}$; that is:

$$Re = C_o + C_1Nu^{1/2} + C_2Nu + C_3Nu^{3/2} + C_4Nu^2 \quad (11)$$

The Reynolds number is evaluated at the gas temperature while the Nusselt number is evaluated at the film temperature ($T_f = \frac{T_w + T_a}{2}$); that is $\nu = \nu_a$ while $k = k_f$. (Note that $Nu^{1/2} \propto E$ for fixed temperature, thus equation 11 reduces to equation 9 for this case.)

An example of a single wire calibration (DISA type 55P76 gold-plated 5 µm wire) is shown in figure (1). A second example is the x-wire (DISA type 55P gold-plated 2.5 µm wires) data shown in figure (2). Both of these calibrations were carried out for conditions corresponding to those present in buoyant plume experiments where the local flow temperature varied over 20°C and where flow velocities ranged from 0.12-1.5 m/s.

The curves shown in figures (1) and (2) were obtained by a linear regression fit to the calibration data. Generally, it was possible to achieve a maximum relative deviation (between predicted and measured velocity) of 0.5% which was well within the accuracy of the velocity measurement in the calibrator. Note that it is important in most cases to minimize the relative error in the curve fitting, and not the absolute error, since otherwise large errors can result at the lower velocities.

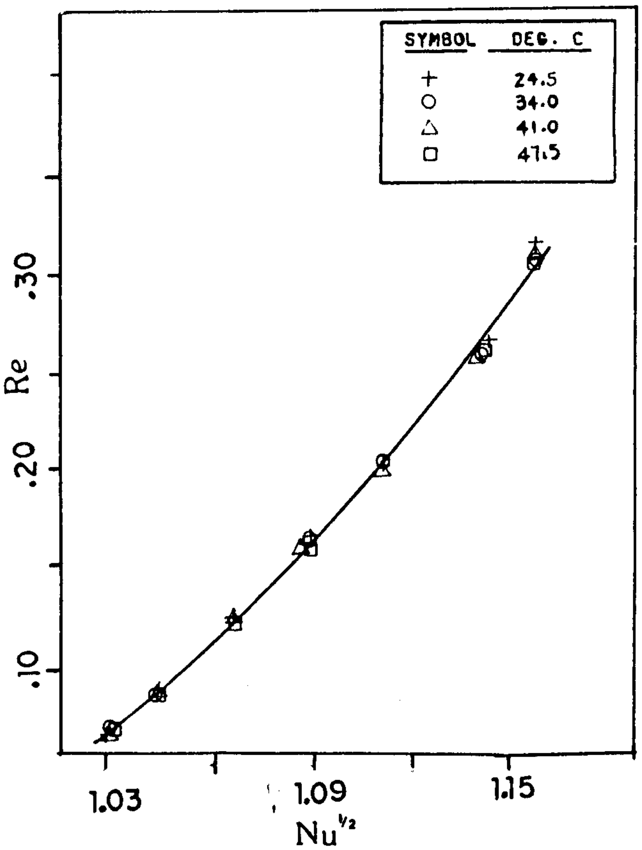


Figure 1. Calibration curve for the DISA 55P76 probe.

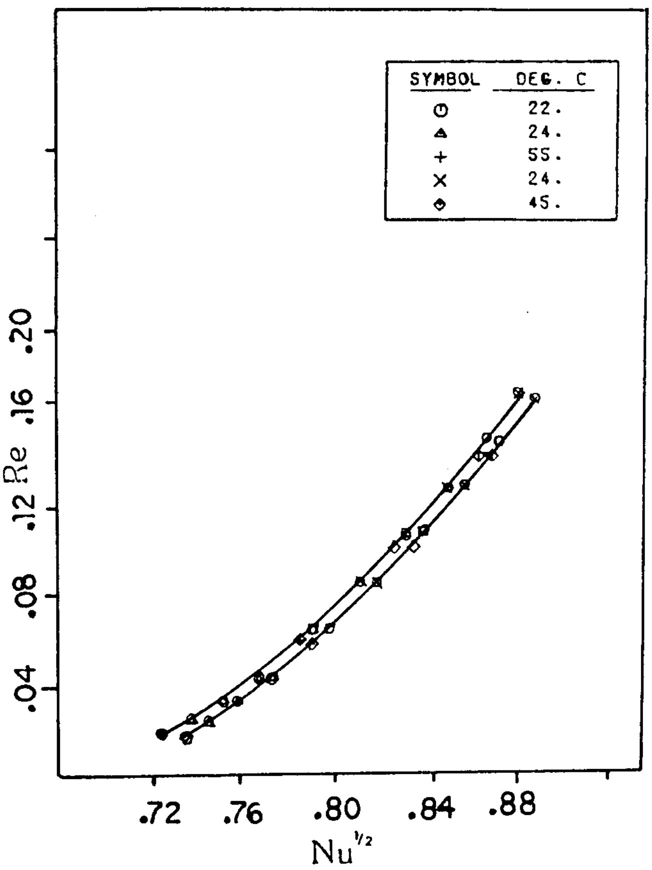


Figure 2. Calibration curve for the DISA 55P gold-plated probe.

Because of slight differences in length and diameter, the calibrations for the two wires do not coincide. These parameters were not measured directly and there was undoubtedly some deviation from the manufacturers stated values. Since the wires are calibrated individually, this is not important, although it would be if a general result (as for a wire of infinite length) were desired. Two parameters which were difficult to measure directly were the sensor resistance and the coefficient of thermal resistivity (and hence the exact wire temperature), the former, since there was no sure way to short the probe without breaking it, and the latter because of a slight dependence on annealing history. Therefore these parameters were adjusted for each wire to give the best collapse of the data at all temperatures. A typical variation was less than a five percent.

While there may be other polynomial expressions which can be used successfully, the use of polynomials in $Nu^{1/2}$ is particularly convenient for digital analysis since $Nu^{1/2} \sim E$, the anemometer bridge voltage. The kinematic viscosity and the thermal conductivity for air are conveniently calculated from the empirical expressions

$$\frac{\nu}{\nu_{ref}} = \left(\frac{T}{T_{ref}}\right)^{1.7} \quad (12)$$

and

$$\frac{k}{k_{ref}} = \left(\frac{T}{T_{ref}}\right)^{0.7} \quad (13)$$

or alternate expressions. The wire temperature $T_w$ is fixed by the feedback amplifier (constant temperature mode) and $T_a$ can be separately measured by a parallel resistance wire, thermocouple, or thermistor. Note that care must be taken in the probe design to ensure that the thermal sensor is not contaminated by the thermal wake of the velocity wires, and yet that all the sensors are close enough to be measuring the same fluctuation.

In the experiments of Beuther (1980) and Shabbir (1987) using x-wires, both the ambient temperature and the velocity anemometer bridge voltages were sampled simultaneously and equation (11) was used to calculate the instantaneous cooling velocities on each wire. This information was subsequently combined with the angle calibration to yield the two desired instantaneous velocity components. By first calculating $Nu^{1/2}$, and then simply using recursive multiplication, the calculation could be rapidly performed.

The effect of including a temperature loading factor in the Nusselt number, i.e.

$$Re = \sum_{n=o}^{4} C_n\left[Nu\left(\frac{T_w}{T_f}\right)^{0.17}\right]^{n/2} \quad (14)$$

has also been investigated. This approach was not as successful as the simpler and more direct approach outlined above, and is not recommended.

## A Quasi-Linear Technique

Over the past forty years it has been rather common to avoid analog linearization entirely and work only with averaged signals. Such a procedure is called quasi-linearization and is justified by expanding equation (6) about average values. Thus,

$$\overline{(E+e)^2} = E^2\left(1 + \frac{\overline{e^2}}{E^2}\right) = A + B\overline{(U+u)^n}$$

$$\simeq A + BU^n\left[1 + \frac{n(n-1)}{2!}\frac{\overline{u^2}}{U^2}\right] \quad (15)$$

where terms above $\overline{u^2}/U^2$ have been neglected. By subtracting this equation from the instantaneous equation, squaring and averaging, expressions for U and $\overline{u^2}$ in terms of E and $\overline{e^2}$ can be obtained (c.f. Corrsin and Uberoi 1949, Rao and Brzustowski 1969). Such expressions are, of course, valid only as long as $\overline{u^2}/U^2$ is truly negligible; in other words, only in low intensity turbulent flows.

Unfortunately for many flows of interest $\overline{u^2}/U^2$ is not negligible--for example, the turbulent jet--and the only recourse is to linearize before averaging. The polynomial calibration scheme proposed above opens a new possibility for quasi-linearization which does <u>not</u> require that powers of the turbulence intensity remain small, but only the less restrictive condition that powers of $\overline{e^2}/E^2$ remain small. This represents a substantial improvement because typically

$$\frac{\overline{e^2}}{E^2} << \frac{\overline{u^2}}{U^2} \quad (16)$$

This is because of the anemometer offset voltage $E_o$ which is included in E is usually much larger than $\overline{e^2}$.

From equation (9) for a constant temperature flow, it follows from decomposing E and U into mean and fluctuating parts and averaging that

$$\bar{U} = A_o + A_1\bar{E} + A_2(\bar{E}^2 + \overline{e^2}) + A_3(\bar{E}^2 + 3\bar{E}\overline{e^2} + \overline{e^3})$$

$$+ A_4(\bar{E}^4 + 6\bar{E}^2\overline{e^2} + 4\bar{E}\overline{e^3} + \overline{e^4}) \quad (17)$$

By subtracting this from equation 9, an equation for the fluctuating velocity can be obtained as:

$$u_1 = A_1 e + A_2\left[2\bar{E}e + (e^2 - \overline{e^2})\right] + A_3\left[3\bar{E}^2 e + 3E(e^2 - \overline{e^2}) + (e^2 - \overline{e^2})\right]$$

$$+ A_4\left[6\bar{E}^2(e^2 - \overline{e^2}) - 4E^2 e + 4E(e^3 - \overline{e^3}) + (e^4 - \overline{e^4})\right] \quad (18)$$

If we assume all terms above the second order to be negligible, equation (17) for the mean velocity reduces to

$$\bar{U} \cong A_o + A_1\bar{E} + A_2\bar{E}^2(1 + \overline{e^2}/\bar{E}^2) + A_3\bar{E}^3(1 + 3\overline{e^2}/\bar{E}^2)$$

$$+ A_4\bar{E}^4(1+6\overline{e^2}/\bar{E}^2) \tag{19}$$

Typically in the experiments described later, $E \simeq 3.4$v while $e_{rms} \simeq 100$ mv, so the neglected terms are at most 3% of the second order terms which are themselves less than 1% of zeroth order terms and could also be neglected.

By squaring and averaging equation (18) for the fluctuating velocity, and again ignoring terms of order higher than the second, the mean square fluctuating velocity can be shown to be given by

$$\overline{u^2} \simeq \left[A_1 + A_2(2\bar{E}) + A_3(3\bar{E}^2) + A_4(4\bar{E}^3)\right]\overline{e^2} \tag{20}$$

Additional terms in $\overline{e^3}/E^3$ could be retained; however, in many applications this seems unnecessary as this quantity is almost always small compared to $\overline{e^2}/E^2$.

Equations (19) and (20) make it clear that accurate measurements of the mean and rms turbulent velocities can be made with the quasi-linearization technique by measuring only the mean and rms anemometer voltages. This is true even in flows of relatively high turbulence intensities ($u'/\bar{u} \sim 1$), because of the relatively low value of $e'/E$. Thus, the polynomial linearization scheme possesses significant advantages when quasi-linearization techniques must be used.

<u>Quasi-linearization in the Presence of Temperature Fluctuations</u>

The same ideas expressed above can be applied to the general heat transfer law, equation (11), to yield a quasi-linear result which is valid, even when the temperature is also fluctuating. This follows by averaging equation (11) to obtain

$$\overline{Re} = B_o + B_1\overline{Nu^{1/2}} + B_2\overline{Nu} + B_3\overline{Nu^{3/2}} + B_4\overline{Nu^2} \tag{21}$$

If we ignore the temperature dependence of the fluid properties and restrict ourselves to only modest temperature fluctuations (relative to the absolute temperature, $\simeq 300^oK$ typically), we can write the Nusselt number as

$$Nu = \frac{C_1E_u^2}{(T_w-T)} \tag{22}$$

where $C_1$ includes the missing terms from equations (2) and (4), and is assumed constant, and $E_u$ denotes the anemometer output.

Expanding, we have

$$Nu = C_1\frac{(\bar{E}_u+e_u)^2}{(T_w-\bar{T})\left[1-(\frac{t}{T_w-\bar{T}})\right]} \tag{23}$$

where t represents the fluctuating temperature. Note that the average wire overheat ratio is given by $(T_w-\bar{T})/\bar{T}$ and is normally greater than 0.5.

By expanding the denominator in a binomial expansion, and neglecting all terms of order higher than the second, it is straightforward to show that the terms of equation (21) are represented by the following expression

$$\overline{Nu^{n/2}} \simeq \left(\frac{C_1\bar{E}_u^2}{T_w-\bar{T}}\right)^{n/2}\left[1 + \frac{n(n-1)}{2}\left(\frac{\overline{e_u^2}}{\bar{E}_u^2}\right) + \frac{n^2}{2}\,\frac{\overline{te_u}}{(T_w-\bar{T})\bar{E}_u}\right] \tag{24}$$

from which $\overline{Re}$ can readily be evaluated. The mean velocity can be calculated immediately from

$$\bar{U} = \frac{\nu}{d}\,\{\overline{Re}\} \tag{25}$$

The expression for $\overline{u^2}$ is given by

$$\begin{aligned}\overline{u^2} = \frac{\nu}{d}\Big[&B_1^2(\overline{Nu} - \overline{Nu^{1/2}}^2) + B_2^2(\overline{Nu^2} - \overline{Nu}^2) + \\ &B_3^2(\overline{Nu^3} - \overline{Nu^{3/2}}^2) + B_4^2(\overline{Nu^4} - \overline{Nu^2}^2) + \\ &2B_1B_2(\overline{Nu^{3/2}} - \overline{Nu^{1/2}}\,\overline{Nu}) + 2B_1B_3(\overline{Nu^2} - \overline{Nu^{1/2}}\,\overline{Nu^{3/2}}) + \\ &2B_1B_4(\overline{Nu^{5/2}} - \overline{Nu^{1/2}}\,\overline{Nu^2}) + 2B_2B_3(\overline{Nu^{5/2}} - \overline{Nu}\,\overline{Nu^{3/2}}) + \\ &2B_2B_4(\overline{Nu^3} - \overline{Nu}\,\overline{Nu^2}) + 2B_3B_4(\overline{Nu^{7/2}} - \overline{Nu^{3/2}}\,\overline{Nu^2})\Big]\end{aligned} \tag{26}$$

Similarly $\overline{ut}$ is given by

$$\overline{ut} = \frac{\nu}{d}\left[B_1\overline{Nu^{1/2}t} + B_2\overline{Nut} + B_3\overline{Nu^{3/2}t} + B_4\overline{Nu^2t}\right] \tag{27}$$

Thus a single anemometer used in conjunction with a fast thermometer (e.g., a resistance wire), D.C. and rms voltmeters, and either a multiplication circuit or summing or differencing amplifiers can yield accurate measurement of not only $\bar{U}$ and $\overline{u^2}$ but also $\bar{T}$, $\overline{t^2}$ and $\overline{ut}$, even when the turbulence intensity is relatively high. A procedure for doing this which was implemented by Ahmed (1980).

Table I shows an evaluation of the quasi-linearization based on (11). The results were obtained in a buoyant plume by direct linearization of the data using

| $(\frac{u'}{\bar{U}})$ | $\bar{U}$(m/s) Digital Tech. | Quasi-Linearization | % Error | $\overline{ut}$ Digital Tech. | Quasi-Linearization | % Error |
|---|---|---|---|---|---|---|
| .336 | .7650 | .7259 | -5.10 | 1.145 | 1.091 | -4.41 |
| .330 | .7468 | .7185 | -6.05 | 0.881 | 0.824 | -6.46 |
| .476 | .558 | .518 | -7.16 | 1.398 | 1.282 | -8.29 |
| .652 | .342 | .328 | -4.10 | 0.883 | 0.776 | -12.1 |
| .940 | .115 | .1086 | -5.50 | 0.174 | 0.128 | -26.0 |

Table I. Comparison of quasi-linearization results with those computed from instantaneous measurements in a turbulent plane.

equation (11), and by quasi-linearization. Even the second order moments are seen to be accurately measured for all but the highest turbulence intensities, a real surprise considering the total disregard of the third moments.

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