Examination of a LSE/POD complementary technique using single and multi-time information in the axisymmetric shear layer

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Abstract

The data base measured by Citriniti (1996) is used to examine the accuracy of using Linear Stochastic Estimation to estimate the velocity field in the annular mixing layer. In the first case, the velocity information is estimated using information from a single instant in time on two radii. In the second approach, the signal is Fourier transformed in time and the frequency dependent coefficients are estimated using coefficients measured on two radii, thus effectively incorporating information from all points in time into the estimate. The estimated velocity fields are then used with the Proper Orthogonal Decomposition to construct low-dimensional models of the flow. It is found that both estimated fields are capable of accurately reproducing the gross features of both the ring structures and the inter-ring regions found in the shear layer, however, the single-time method does not accurately reproduce the amplitude of the fluctuating velocity.

1 Introduction

It is widely recognized that large energetic structures play an important role in the dynamics of the near field in many free-shear flows, such as the axisymmetric shear layer occurring near the exit of a round jet. Although there are many different techniques that can be used to identify the coherent structures (*cf* Bonnet and Glauser 1994), only a few of these can be used to study the dynamics of the structures. One technique that can be used for this purpose is the Proper Orthogonal Decomposition (POD) technique introduced by Lumley (1967).

One of the primary difficulties of studying the *instantaneous* dynamics of the large structures with this technique (and others) is that it requires the simultaneous measurement of the velocity over the region of interest. As a result, there have been few attempts to experimentally study the instantaneous dynamics of the structures on an entire plane in a free-shear flow. In one attempt to investigate these features Citriniti (1996) simultaneously measured the velocity field at 138 points in a high-Reynolds-number annular shear layer using hot-wire anemometers. Using the POD technique he found evidence that the shear layer was made up of both ring vortices and streamwise rib vortices that occur between the vortex rings.

Although this experiment yielded very useful physical information about the axisymmetric shear layer, the effort required to perform the experiment effectively precludes its use on a wide variety of flows. One alternative approach suggested by Ukeiley *et al.* (1993) is to measure the velocity at a small number of points and use Linear Stochastic Estimation

(LSE) to approximate velocity over the rest of the field. This estimated velocity field could then be used as an input into POD technique and an estimate for the dynamics of the large structures could be deduced. Ukeiley *et al.* (1993) and Bonnet *et al.* (1994) demonstrated that this technique was capable of accurately reproducing the dynamics of the large structures when applied to one non-homogeneous direction in a shear layer. They also suggested that the method could be extended to more spatial dimensions in a flow however, as yet, there has not been an attempt to confirm this.

The objective of this investigation is to examine if LSE can be used in conjunction with the POD technique to accurately estimate the dynamics of the large structures on a full $r-\theta$ plane in the axisymmetric shear layer. Here, two different approaches are used. In the first approach, only the information from a single point in time are used to estimate the velocity field on the plane at the same instant. In the second approach, the LSE is applied to the Fourier coefficients for the field (transformed in time) thus incorporating information from all points in time. In order to determine how accurately these estimated fields reproduce the dynamics of the large-scale motions, low-order models for the flows dynamics are computed by applying the POD to the estimated fields. These models are then compared to the analogous model computed from the actual velocity field (Citriniti 1996).

2 Background

2.1 Annular Shear Layer Data Base

In order to study the dynamics of the large structures in the axisymmetric shear layer Citriniti (1996)[2] constructed an array of 138 single-wire transducers that was placed at a distance of 3 diameters downstream of a round jet. Glauser (1987) had shown previously that the large structures in flow could be resolved using the POD if the velocity was measured at 6 radial positions across the layer at $r/d = 0.15, 0.28, 0.41, 0.54, 0.67$, and 0.80. He also found that the azimuthal dependence of the structures could be recovered if the velocity was measured at 6 azimuthal positions at the inner most radial position and 12, 24, 32, 32, and 32 positions at the other 5 radial positions, respectively. This information was used to locate the probes within the 138 wire array. The mean velocity normalized by the exit velocity ranged from 0.9 to 0.15 over these 6 radial points indicating that the array of hot wires measured the streamwise velocity component in a region that spanned most of the shear layer.

The exit velocity profile of the jet was a top hat to within 0.1% with a turbulence intensity of 0.35%. The outlet velocity of the jet was set such that the exit Reynolds number based on the nozzle diameter, d, was 80,000. For these conditions, Glauser (1987) demonstrated that the large structures could be well resolved in the first 800 Hz of the spectra so the signals from all of the wires in the array were filtered at 800 Hz and simultaneously sampled at 2,048 Hz. The data was gathered in 300 blocks of 1024 points, which allowed the accurate computation of spectral information.

2.2 Proper Orthogonal Decomposition

Lumley (1967) argued that the functions used to represent the large structures in a turbulent flow should be defined in an objective manner using information from the flow only. Lumley suggested these functions could be defined as those that make the largest contribution to

[2] See also the paper by Citriniti in this volume.

the turbulent energy in the region of interest. Using this definition, it can be shown that functions for homogeneous and stationary directions are simply Fourier modes so these directions can be transformed out of the problem in the standard manner.

For the axisymmetric shear layer, the functions that satisfy Lumley's definition for the radial direction, $\Phi^n(r,m,f)$, are solutions to the integral eigenvalue problem given by (Citriniti 1996)

$$\int_0^\infty F(r,r',m,f)\Phi^n(r',m,f)r'dr' = \lambda^n(m,f)\Phi^n(r,m,f), \tag{1}$$

where $F(r,r',m,f)$ is the Fourier transform of the two-point velocity correlation in both the azimuthal direction and time, m is the azimuthal mode number and f is the frequency. It is straightforward to show that these functions are orthogonal (*cf* Lumley 1967). Thus, the integral eigenvalue can be solved using only statistical measures of the flow to yield an orthogonal basis to optimally describe the flow in the radial direction.

The information about the dynamics of the motions in the annular shear layer can then be determined by computing the coefficients for the orthogonal functions, which are given by

$$a^n(m,f) = \int_0^\infty \hat{u}(r,m,f)\Phi^{n*}(r,m,f)rdr, \tag{2}$$

where $\hat{u}(r,m,f)$ is the Fourier transform of the *instantaneous* velocity field in the azimuthal direction and time. Thus, the coefficients can only be determined if the velocity field is simultaneously sampled over the region of interest.

The dynamics of the large-scale motions can then be studied by reconstructing the Fourier coefficients using a small number of modes; *i.e.*,

$$\hat{u}_{rec}(r,m,f) = \sum_{n=1}^{N} a^n(m,f)\Phi^n(r,m,f), \tag{3}$$

where N is a user specified parameter. If N is small then only the information from the most energetic modes is retained, thus reducing the small-scale information in the signal. The amount of small-scale information in the reconstructed velocity field can be further reduced by including the information from only a small number of azimuthal modes. The coefficients for the other modes are set to zero in order to remove them from the reconstructed velocity. This filtered signal can then be inverse Fourier transformed in time and the azimuthal direction to yield a low-order model for the instantaneous velocity field in the axisymmetric shear layer.

Citriniti (1996) demonstrated that the dynamics of the large-scale structures could be modeled if only the first POD mode is used (*i.e.*, N=1) and only the coefficients for azimuthal mode numbers 0 and $3-6$ are retained in the reconstructed field. Examples of the reconstructed instantaneous fluctuating velocity fields computed using these modes are shown in figure 1. In both of these figures the grey scale corresponds to the value of the instantaneous fluctuating velocity. The light colors correspond to positive fluctuating streamwise velocities while the dark colors correspond to negative fluctuating streamwise velocities.

It is clear that there is a band of faster-than-average moving fluid inside a band of slower-than-average moving fluid in figure 1(a), which is consistent with the pattern one would expect with the passage of a vortex ring like structure. On the other hand, slightly after the passage of the 'ring' there are neighbouring regions of fast and slow moving fluid (*v* figure 1(b)), suggesting that regions of fast flow moving out of the core neighbour regions of slow moving fluid being entrained into the core. This is consistent with the pattern one would expect in a braid region of counter-rotating streamwise vortices. Thus, the velocity

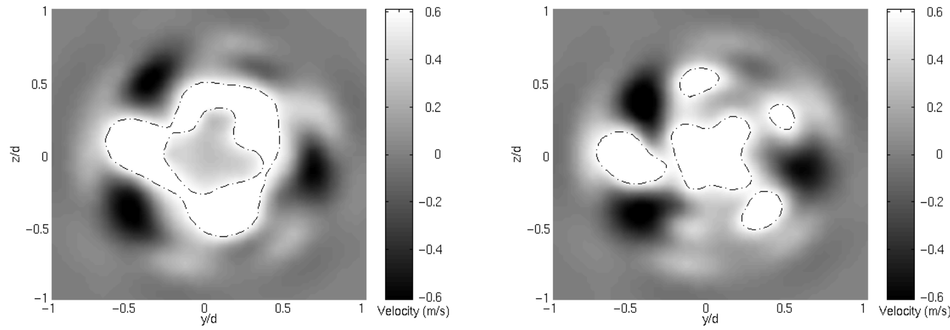


Figure 1: Partial reconstructions of the actual velocity field using the first POD mode and azimuthal modes $0, 3, 4, 5$, and 6. (a) ring-like structure (b) inter-ring region that appears to contain a series of counter-rotating streamwise vortices.

field reconstructed with only a few modes on the $r - \theta$ plane show evidence of both ring vortices and streamwise vortices in the inter-ring regions.

2.3 Linear Stochastic Estimation

Although these results shed insight into the dynamics of the structures in the axisymmetric shear layer, it is not trivial to simultaneously measure the velocity field at 138 points in the field. Thus, as Ukeiley *et al.* (1993) suggested, it would be useful if a similar picture of this flow (and others) could be deduced by applying the POD to an instantaneous field that was measured at a smaller number of points and estimated at the other points necessary to resolve the POD coefficients. Ukeiley *et al.* suggested using Linear Stochastic Estimation to extrapolate information from the measured points and fill out the field sufficiently to use the POD.

2.3.1 Single Time Estimation

For the current application, it is useful to estimate the Fourier coefficients of the velocity field, such as $\hat{u}(r_n, m, t)$, using Fourier coefficients measured at one or two other radial positions. For example, if the Fourier coefficients are measured at two radial positions, r' and r'', it follows that the coefficients at the other radial positions can be approximated as

$$\hat{u}_e(r_n, m, t) = A(r_n, m)\hat{u}(r', m, t) + B(r_n, m)\hat{u}(r'', m, t), \tag{4}$$

where $\hat{u}(r', m, t)$ and $\hat{u}(r'', m, t)$ are the measured Fourier coefficients at r' and r''. In this case, the mean square errors in the estimated coefficients, $E\{(\hat{u} - \hat{u}_e)(\hat{u} - \hat{u}_e)^*\}$, are minimized if

$$A(r_n, m) = \frac{\phi(r_n, r', m)\phi(r'', r'', m) - \phi(r_n, r'', m)\phi(r'', r', m)}{\phi(r'', r'', m)\phi(r', r', m) - |\phi(r'', r', m)|^2} \tag{5}$$

and

$$B(r_n, m) = \frac{\phi(r_n, r'', m)\phi(r', r', m) - \phi(r_n, r', m)\phi(r', r'', m)}{\phi(r'', r'', m)\phi(r', r', m) - |\phi(r'', r', m)|^2}, \tag{6}$$

where $\phi(r_n, r', m) = \overline{\hat{u}(r_n, m, t)\hat{u}^*(r', m, t)}$. The LSE coefficients, $A(r_n, m)$ and $B(r_n, m)$, are independent of time because the flow is stationary so that the same coefficients are used for both the ring and inter-ring regions of the flow.

It is also clear that the amplitude of the LSE coefficients is essentially determined by the correlation of the signals between the estimated point and the measured points. One difficulty of using the single time approach is that it combines contributions from both large-scale motions that are likely correlated across the layer and small-scale motions that are essentially uncorrelated since

$$\phi(r_n, r', m) = \int F(r_n, r', m, f) df. \tag{7}$$

Thus, the LSE coefficient may underpredict the large-scale motions because their contribution has been biased down by the contribution of the uncorrelated small-scale motions.

2.3.2 Frequency Mode Estimation

The problem can be reduced by incorporating information from different times thus shifting the weighting towards the large-scale motions that are correlated over longer times. This, however, has a tendency to increase the complexity of the expression for the estimated velocity and thus the equation set for the LSE coefficients. One technique to retain the time information efficiently is to Fourier transform the time signals and use LSE to approximate the individual Fourier coefficients; *i.e.*,

$$\hat{\hat{u}}_e(r_n, m, f) = C(r_n, m, f)\hat{\hat{u}}(r', m, f) + D(r_n, m, f)\hat{\hat{u}}(r'', m, f), \tag{8}$$

where $\hat{\hat{u}}(r', m, f)$ and $\hat{\hat{u}}(r'', m, f)$ are the measured Fourier coefficients at two radial positions, r' and r''. The mean square error in these coefficients are minimized if

$$C(r_n, m, f) = \frac{F(r_n, r', m, f)F(r'', r'', m, f) - F(r_n, r'', m, f)F(r'', r', m, f)}{F(r'', r'', m, f)F(r', r', m, f) - |F(r'', r', m, f)|^2} \tag{9}$$

and

$$D(r_n, m, f) = \frac{F(r_n, r'', m, f)F(r', r', m, f) - F(r_n, r', m, f)F(r', r'', m, f)}{F(r'', r'', m, f)F(r', r', m, f) - |F(r'', r', m, f)|^2}. \tag{10}$$

where $F(r_n, r', m, f) = \overline{\hat{\hat{u}}(r_n, m, f)\hat{\hat{u}}^*(r', m, f)}$. These coefficients are determined individually for each frequency and should be more capable of accurately extrapolating the information from the large scale motions.

2.4 Combining the LSE and POD

In the axisymmetric shear layer, the LSE technique could be use to interpolate or extrapolate information in two different spatial directions, r or θ. Since the signals are correlated over a greater distance in the radial direction than the azimuthal direction (relative to the characteristic length scales in those directions) it follows that the LSE should work more effectively in the radial direction than in the azimuthal direction. For this reason the application of the LSE to this direction is considered first. The application of this technique to the azimuthal direction will be reported elsewhere.

In all the cases considered here it is assumed that the Fourier coefficients $\hat{u}(r, m, t)$ or $\hat{\hat{u}}(r, m, f)$ are known at one or two radial positions in the annular shear layer. This information is then used to estimate the Fourier coefficients at the other radial positions. Physically, this corresponds to doing the experiment carried out by Citriniti (1996) using only one or two rings of hot-wires instead of the six used in that experiment.

The resulting estimated velocity fields can then be projected onto the orthogonal basis deduced by Citriniti (1996) in order to determine estimates of the POD coefficients; *i.e.*,

$$a_e^n(m,f) = \int_0^\infty \hat{\hat{u}}_e(r,m,f)\Phi^{n*}(r,m,f)rdr, \tag{11}$$

where $\hat{\hat{u}}_e(r,m,f)$ is the double Fourier transform of the velocity field. (Here, of course, it is necessary to solve a discrete version of this equation; *cf* Citriniti 1996.) This estimated Fourier coefficient is computed directly in the second method outlined above. In the first case, where the Fourier coefficient $\hat{u}_e(r_n,m,t)$ is estimated at each time, it is necessary to Fourier transform the resulting Fourier coefficient in time. These estimated POD coefficients can then be used to partially reconstruct the Fourier coefficients (eq. 3), which can then be inverse Fourier transformed to construct a low-dimensional model for the velocity field at all the spatial points in the field.

It is interesting to note that in this process it is actually more desirable to accurately estimate the POD coefficients (particularly the first one) than it is too accurately estimate the Fourier coefficient for the field. Thus, it might be more logical to argue that the coefficients for the LSE should be chosen to minimize the mean square error in the estimated POD coefficient; *i.e.*, $E\{(a^n - a_e^n)(a^n - a_e^n)^*\}$ where $*$ denotes a complex conjugate. It can be shown that this error is, in fact, minimized when the frequency Fourier coefficient $\hat{\hat{u}}(r_n,m,f)$ are estimated using the LSE (eq. 8-10).

3 Results

Initially the suitability of different configurations was examined by considering the normalized means square error in the estimated coefficients; *i.e.*,

$$\frac{\overline{|(\hat{u}_e(r_n,m,t) - \hat{u}(r_n,m,t))|^2}}{\phi(r_n,r_n,m)} \text{ or } \frac{\overline{|(\hat{\hat{u}}_e(r_n,m,f) - \hat{\hat{u}}(r_n,m,f))|^2}}{F(r_n,r_n,m,f)}.$$

It was found that no single radial positions could be used to accurately extrapolate the Fourier coefficients for both azimuthal mode 0 and modes $3-6$. On the other hand it was found that all the modes could be reasonable well approximated if the Fourier coefficients from two radii on opposite sides of the shear layer were used to estimate the coefficients at the other radii. The lowest error seemed to occur when the two radial positions were chosen near center of the shear layer, at $r/d = 0.41$ and $r/d = 0.67$ (analogous to the result outlined by Bonnet *et al.*). These two points (the third and fifth radial positions) were used for all the results presented here.

The normalized error in the estimated Fourier coefficients for the case where information from a single point in time are used to estimate $\hat{u}(r_\alpha,m,t)$ are shown in figure 2. The error is 0 at the two measured points but is quite large for all the radii. For example, even near the center, where the ring mode is dominant, the normalized error in the coefficients for mode 0 is approximately 40% while the error in the higher azimuthal modes is approximately 60% even at the outer radii where they play a dominant role. These levels of error, although high, are not inconsistent with the level of error that occur in other applications of the LSE to shear layers (*e.g.* Bonnet *et al.* 1994).

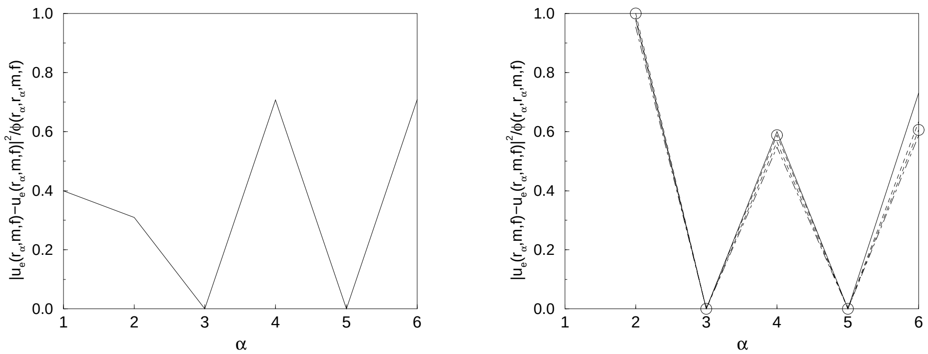


Figure 2: Normalized mean square error in the estimated Fourier coefficients at the points $r_\alpha/d = 0.15, 0.28, 0.41,\ 0.54, 0.67$, and 0.80 using information from a single point in time. (a) azimuthal mode 0 (b) azimuthal modes —— 3, ---- 4, —-— 5, and ---- ∘ 6.

On the other hand, examples of the the normalized mean square error in the Fourier coefficient $\hat{\hat{u}}_e(r_n, m, f)$ determined by directly estimating the Fourier coefficients are shown in figure 3. It is clear that the mean square errors for azimuthal mode 0 are small ($\sim 20\%$) for a range of frequencies around 100 Hz for all the radial positions except the fourth radial position. This corresponds to the peak in the mode 0 spectrum at the Strouhal frequency so the error in the coefficients are small over the energy containing range of the spectrum. It is not clear why the LSE performs more poorly at the central point but a similar result was noted by Bonnet *et al.* (1994).

Similarly the normalized error in the azimuthal mode 5 coefficients are shown in figure 3(b) (the errors for modes 3, 4 and 6 are very similar). It is important to note that the spectrum for this mode peaks at the lowest frequency (*cf* Citriniti 1996). Thus, the normalized mean square error is a minimum at the peak in the spectra. Although this error is larger than the error at the peak of mode 0 spectrum, it is smaller than the error for the higher modes for the single time case suggesting that the direct estimate of the frequency Fourier coefficients should provide a more accurate estimate of the dynamics in the inter-ring region.

Of course, the most important measure of these techniques is not the mean square error, but rather how accurately they reproduce the dynamics of the large-scale motions in the annular shear layer. In order to examine this question, a partial reconstruction of the field was computed using the estimated velocity fields from both techniques. Following the approach outlined by Citriniti (1996), only the first POD mode and only azimuthal modes 0, 3, 4, 5, and 6 were retained in the reconstruction. A realization from each of these reconstructions and the reconstruction from the original field are shown in figure 4. All three realizations are taken from the same point in time during the passage of a ring like structure. It is clear that the velocity field estimated from information at a single time yields a reconstructed velocity field that is much smaller in magnitude than the actual field ($\sim 50\%$). A similar level of error was found using realizations from the inter-ring region. On the other hand, the reconstructed velocity field computed using the field estimated by the frequency method more closely approximates the magnitude of the actual velocity field. Thus, as expected, the estimation in the frequency domain more accurately predicts the

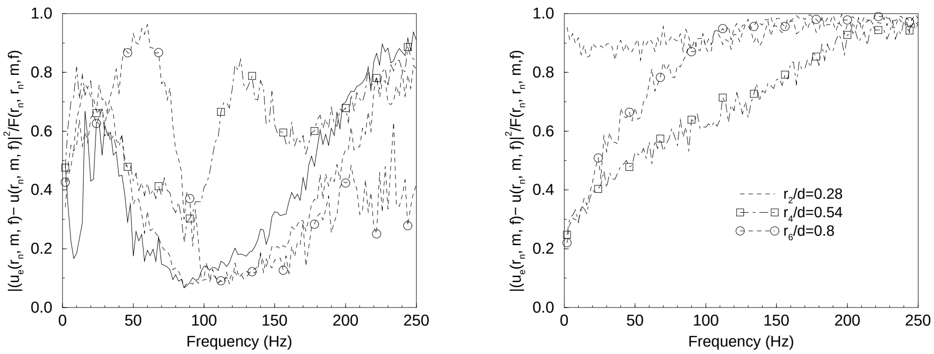


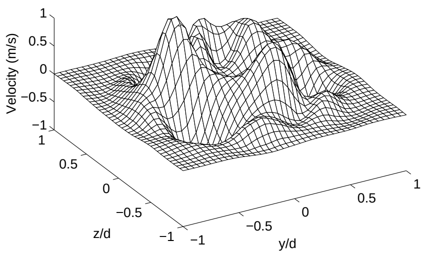
Figure 3: Normalized mean square error in the estimated Fourier in the estimated Fourier coefficients at the points —— $r_1/d = 0.15$, ---- $r_2/d = 0.28$, —-— $r_4/d = 0.54$, and ---- ∘ $r_5/d = 0.8$ (a) azimuthal mode 0 (b) azimuthal mode 5.

amplitude of the actual velocity signal.

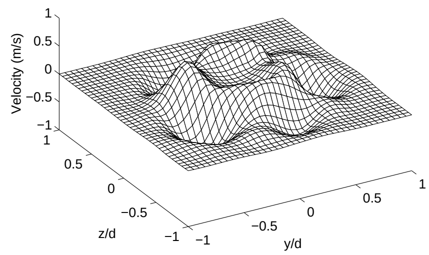
In order to examine how accurately the two techniques reproduce the topology of the structures it is more useful to look at a two-dimensional contour plot of the instantaneous fluctuating velocity on the $r - \theta$ plane computed using both the estimated fields and the actual field . The estimated and actual reconstructed fields for realizations showing the ring region and the inter-ring region are shown in figure 5. The grey scale again corresponds to the fluctuating velocity but note that here the range of the grey scale for the single-time estimation technique has been modified in order to facilitate the comparison of the structures topology. Somewhat surprisingly the reconstructed fields from both the estimated fields have very similar topologies. It is also clear that the topology of the structure in the actual field is well represented by both the estimated fields. The estimated fields do not, however, capture all of the fine scale information of the structure as the estimation technique effectively acts as a filter.

4 Summary

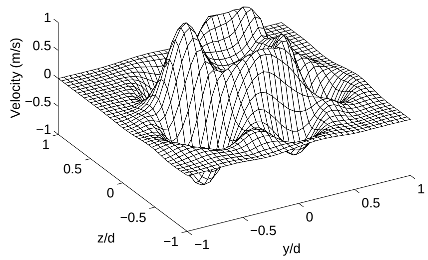
The accuracy of using estimated velocity fields to study the dynamics of the large structures in the axisymmetric shear layer was examined using the data base measured by Citriniti (1996). It was found that the topology of the ring structures and inter-ring regions could be well reproduced if the Fourier coefficients were measured on 2 radii and estimated on 4 others. This was true whether the information from only a single time or all times was incorporated into the estimation process. It was found however that the amplitude of the fluctuating velocities in the structures were not well recovered when only information from a single time was used. The magnitude of the velocity field could be accurately predicted if the estimation was carried out on the Fourier transform of the signals in time thus effectively incorporating information from all times in the estimate of the velocity field. This process reduced the number of probes necessary to carry out the experiment from 138 to 56 thus effecting a 60 percent savings in the number of probes. Interpolation in the azimuthal direction will be considered in the future.



(a) original signal



(b) single-time method



(c) frequency method

Figure 4: Partial reconstruction of the velocity field using the first POD and azimuthal modes $0, 3, 4, 5$, and 6. Surface height corresponds to the value of the fluctuating velocity.

References

Bonnet, J. P. and Glauser, M. N. (1994). *Eddy Structure Identification in Free Turbulent Shear Flows.* Kluwer Academic Press.

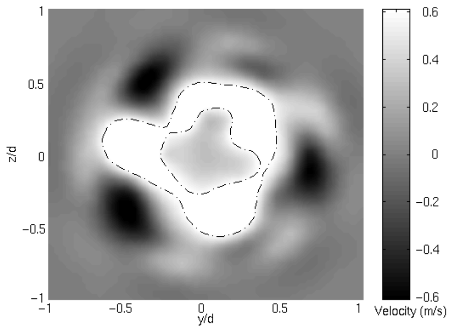
Bonnet, J. P., Cole, D. R., Deville, J., Glauser, M. N., and Ukeiley, L. S. (1994). Stochastic estimation and proper orthogonal decomposition: Complementary techniques for identifying structure. *Expts. in Fluids* **17**, 307–314.

Citriniti, J. (1996), *Experimental investigation into the dynamics of the axisymmetric mixing layer utilizing the Proper Orthogonal Decomposition.* PhD thesis, State University of New York at Buffalo.

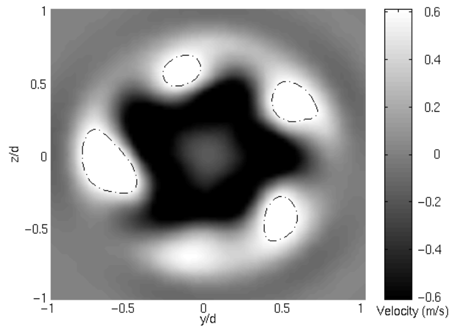
Glauser, M. N. (1987), *Coherent Structures in the Axisymmetric Turbulent Jet Mixing Layer.* PhD thesis, State University of New York at Buffalo.

Lumley, J. L. (1967). The structure of inhomogeneous turbulence. In Yaglom, A. M. and Tatarski, V. I., editors, *Atmospheric Turbulence and Radio Wave Propogation*, pages 166–178. Nauka, Moscow.

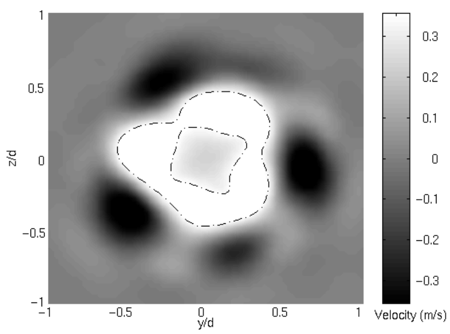
Ukeiley, L. S., Cole, D. R., and Glauser, M. N. (1993). An examination of the asisymmetric jet mixing layer using coherent structure detection techniques. In *Eddy Structure Identification in Free Turbulent Shear Flows.* Kluwer Academic Publishers.



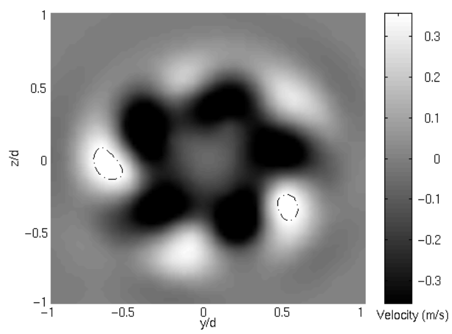
(a) original signal



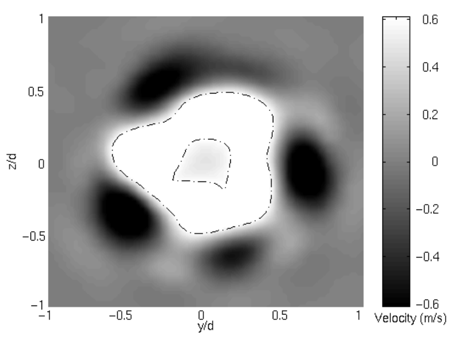
(b) original signal



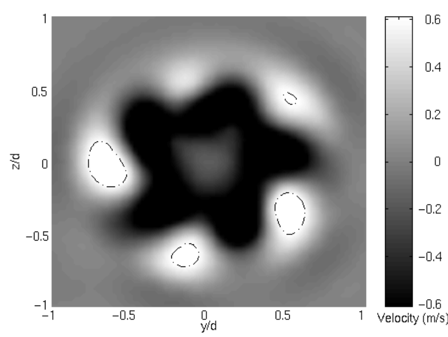
(c) single-time method



(d) single-time method



(e) frequency method



(f) frequency method

Figure 5: Partial reconstruction of the velocity field using the first POD and azimuthal modes $0, 3, 4, 5$, and 6. Grey scale corresponds to the value of the instantaneous velocity in the reconstructed field. Figures (a), (c), and (e) correspond to $\tau = 422$ in Citriniti (1996) while (b), (d), and (e) correspond to $\tau = 431$