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Eddy Viscosity Calculations of Turbulent Buoyant Plumes

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ABSTRACT

It is shown that a simple eddy viscosity model can predict accurately the velocity and temperature profiles of a fully developed axisymmetric buoyant plume in neutral surroundings, if the contribution of the turbulence to the vertical heat transport is accounted for.

Calculations with turbulent Prandtl numbers near unity are shown to give the best agreement with experimentally determined profiles. It is also shown that for flows in local scale equilibrium such as the simple plume, eddy viscosity models can provide a useful means for discriminating between conflicting experimental data.

Finally, reasons for the success of these simple models in buoyant plumes, possible additional flows in which the models can succeed, and probable limitations on this type of modeling are briefly discussed.

NOMENCLATURE

A = width constant defined by equation 29.

F_o = rate at which buoyancy added at source defined by equation 6.

f = modified axial velocity function defined by equation 24.

f_1 = axial velocity function defined by equation 7.

f_o = centerline value of $f_1(\eta)$.

g = gravitational acceleration.

H_s = turbulent heat flux scale defined by equation 8.

h_1 = radial turbulent heat flux function defined by equation 7.

h_2 = axial turbulent heat flux function defined by equation 7.

k_1 = radial velocity function defined by equation 7.

m = exponent occurring in equation 29.

P_T = turbulent Prandtl number defined by equation 17.

Q = fraction of total buoyancy (heat) carried by mean motion equation 23.

R_s = turbulent energy scale defined by equation 8.

R_T = turbulent Reynolds number defined by equation 16.

r = radial coordinate

s_1 = Reynold's stress function defined by equation 7.

T = mean temperature

T_o = value of $t_1(\eta)$ at $\eta = o$, centerline.

T_s = temperature scale defined by equation 8.

t = modified temperature difference function defined by equation 24.

t_1 = temperature difference function defined by equation 7.

t_o = centerline value of $t_1(\eta)$.

U = mean vertical velocity.

U_s = velocity scale defined by equation 8.

u = fluctuating vertical velocity.

V = mean radial velocity.

v = fluctuating radial velocity.

x = vertical, or axial, coordinate.

$\alpha(\eta)$= dimensionless function containing radial dependence of eddy diffusivity.

α_e = eddy diffusivity (equation 2).

β = coefficient of thermal expansion.

η = r/x, dimensionless radial coordinate.

η_1 = integration (or dummy) variable.

$\nu(\eta)$= dimensionless function containing radial dependence of eddy viscosity.

ν_e = eddy viscosity (equation 1).

ρ = ambient fluid density

θ = fluctuating temperature

ξ = dimensionless radial coordinate defined by equation 24.

ξ_1 = integration (or dummy) variable.

ΔT = mean temperature difference from ambient.

INTRODUCTION

Few concepts in turbulence theory are more widely used than the closure of the averaged equations of motion by an eddy viscosity. In the form originally proposed by Boussinesq in 1877 [1], a simple proportionality relationship between turbulent transport and mean gradient is assumed, the constant of proportionality being the eddy viscosity (or diffusivity if concentration or heat are being considered). For example, the turbulent Reynolds stress is written

$$-\overline{uv} = \nu_e \frac{\partial U}{\partial r} \qquad (1)$$

and the turbulent heat flux is taken to be

$$-\overline{v\theta} = \alpha_e \frac{\partial T}{\partial r} \qquad (2)$$

The similarity between the two relationships written above is often referred to as Reynolds' analogy.

The eddy viscosity concept also plays an important role in higher order closure methods of turbulence theory. The so-called "one-equation" and "two-equation" models are simply techniques for generating an eddy viscosity with spatial variation (c.f. Launder and Spalding [2]). Even the turbulence models which deal directly with the averaged equations for the Reynolds stress and turbulent heat flux use eddy viscosity-type terms to model the third-order moment terms for the turbulent diffusion (c.f. Reynolds [3]).

The simple eddy viscosity models which utilize only the mean equations have had a bewildering variety of successes and failures in modeling turbulent flows. Tennekes and Lumley [4] argue that the eddy viscosity can be expected to be successful when the turbulent flow is characterized by single time and length scales. Thus the presence or absence of dynamically important multiple length or time scales can provide useful clues as to whether an eddy viscosity model might be successful. Another way to rephrase the thesis of Tennekes and Lumley is to state that eddy viscosity models might be expected to work when the flow is in local scale equilibrium. It will be shown that the simple buoyant plume represents such a flow; thus an eddy viscosity formed from local parameters, either assumed or calculated, should provide reasonable solutions.

There have been numerous attempts to model both the turbulent buoyant plume and forced jets involving varying amounts of buoyancy. These attempts have ranged from the integral entrainment velocity models of Morton et al [16], [17] and the mixing length model of Madni and Pletcher [18], to the second order turbulence models of Chen and Rodi [19] and Tamanini [20]. It is not the purpose of this paper to evaluate these attempts or even improve upon them. Rather, it is our goal to explore in detail the application of the simple eddy viscosity to a flow which completely satisfies the conditions for its application. Of particular interest will be the model's ability to accurately predict the profiles, it's sensitivity to choice of constants, and the importance of the non-negligible vertical turbulent heat flux. We follow closely the approach of Yih [7] who obtained closed form solutions for turbulent Prandtl numbers of 1.1 and 2.0 by neglecting the vertical turbulent transport

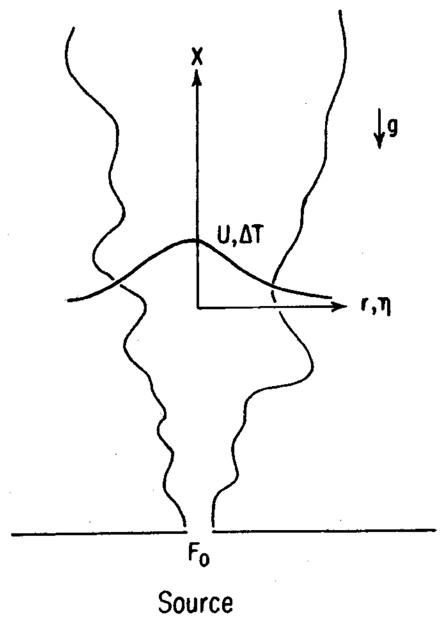

Figure 1 - Sketch showing plume and coordinates.

THE TURBULENT BUOYANT PLUME

The simple turbulent buoyant plume (shown schematically in figure 1) is a useful model of many naturally occurring processes. A vertical column of fluid is driven by a buoyancy source at the base and spreads by turbulent entrainment. The plume is assumed to be turbulent and fully developed, stationary in the mean, and to have a sufficiently high turbulent Reynolds number that viscous terms can be neglected in the equations for the mean flow. Making the usual Boussinesq approximations (c.f. Tennekes and Lumley [4]) we write for the mean motion of axisymmetric flow,

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}(-r\overline{uv}) + g\beta\Delta T \tag{3}$$

$$U\frac{\partial T}{\partial x} + V\frac{\partial T}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}(-r\overline{v\theta}) + \{\frac{\partial}{\partial x}-(\overline{u\theta})\} \tag{4}$$

$$\frac{\partial U}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rV) = 0 \tag{5}$$

ΔT is the difference between the local temperature and the temperature at infinity and β represents the thermal expansion coefficient. The mean pressure gradient term has been eliminated using the r-momentum equation, and the streamwise gradient of the turbulent normal stress gradient has been neglected. The streamwise gradient of the turbulent heat (temperature) flux which is usually neglected has been retained, for now.

For later use we integrate the temperature equation across the flow to obtain

$$2\pi\int_0^\infty g\beta\,[U\Delta T + \overline{u\theta}]r\,dr \equiv F_o \tag{6}$$

where ρF_o can be identified as the rate at which weight deficiency or buoyancy is added to the source. For the neutrally stable environment assumed here, ρF_o is also the rate at which buoyancy crosses any given plane.

It is appropriate to note at this time that the turbulent contribution to the integral above may not be negligible. For the axisymmetric case, the contribution of the turbulence has been estimated as high as 15% (George, Alpert and Tamanini [5]). This question and its significance will be discussed later.

If we assume that the ambient fluid is of uniform density (or neutrally stable), the flow is entirely characterized by the rate at which buoyancy is added at the source and the distance from the source. We seek similarity solutions of the form

$$\begin{aligned} \eta &= r/x \\ U &= U_s f_1(\eta) \\ V &= U_s k_1(\eta) \\ g\beta\Delta T &= T_s t_1(\eta) \\ \overline{uv} &= R_s s_1(\eta) \\ g\beta\overline{vt} &= H_s h_1(\eta) \\ g\beta\overline{ut} &= H_s h_2(\eta) \end{aligned} \tag{7}$$

Substituting into the equations of motion, demanding that the coefficients of terms in the equations be independent of x, and applying the integral condition that F_o be independent of x yield the following functional relationships (c.f. Tennekes and Lumley [4], Rouse, Yih and Humphreys [6]):

$$\begin{aligned} U_s &= F_o^{\frac{1}{3}} x^{-\frac{1}{3}} \\ T_s &= F_o^{\frac{2}{3}} x^{-\frac{5}{3}} \\ R_s &= F_o^{\frac{2}{3}} x^{-\frac{2}{3}} \\ H_s &= F_o x^{-2} \end{aligned} \tag{8}$$

Using these the equations of motion reduce to

$$-\frac{1}{3}f_1^2 - \eta f_1 f_1' + k_1 f_1' = -\frac{1}{\eta}\frac{d}{d\eta}(\eta s_1) + t_1 \tag{9}$$

$$-\frac{5}{3}f_1 t_1 - \eta f_1 t_1' + k_1 t_1' = -\frac{1}{\eta}\frac{d}{d\eta}(\eta h_1) - \{2h_2 + \eta h_2'\} \tag{10}$$

$$-\frac{1}{3}f_1 - \eta f_1' + k_1' + \frac{k_1}{\eta} = 0 \tag{11}$$

The dependence of the set of equations above on k_1 (the cross-stream velocity component) can easily be removed by integrating the continuity equation (eqn 11) and substituting for k_1 in the momentum and temperature equations. The results are

$$-\frac{1}{3}f_1^2 - \frac{5}{3}\frac{f_1'}{\eta}\int_0^\eta \eta_1 f_1\,d\eta_1 = -\frac{1}{\eta}\frac{d}{d\eta}(\eta s_1) + t_1 \tag{12}$$

$$-\frac{5}{3}f_1 t_1 - \frac{5}{3}\frac{t_1'}{\eta}\int_0^\eta \eta_1 f_1\,d\eta_1 = -\frac{1}{\eta}\frac{d}{d\eta}(\eta h_1) - \{2h_2 + \eta h_2'\} \tag{13}$$

THE EDDY VISCOSITY MODEL FOR THE PLUME

It is clear that these equations (and therefore the flow) are completely characterized by single length and time scales since all lengths are proportional to x, and all time scales to x/U_s. Thus from our previous discussion, we expect that an eddy viscosity model will be successful in predicting the evolution of this flow.

On dimensional grounds we must have

$$\nu_e = U_s x\,\nu(\eta) \tag{14}$$

$$\alpha_e = U_s\,x\,\alpha(\eta) \tag{15}$$

There is no reason, of course, to expect that the eddy viscosity should be the same for both axial and cross-stream heat flux. For reasons which shall be presented later in the discussion, it is not necessary to include the axial gradients of the turbulent heat flux in the calculation even though the turbulent fluctuations may contribute a significant fraction of the total heat flux. Therefore we shall ignore the bracketed term of equation (13) which represents the turbulence contribution to the vertical heat transport. (We will later account for this contribution in the integrated energy balance or the buoyancy integral).

Since the flow being modeled is a free shear flow, it may be assumed well-mixed since its dynamics will be dominated by a single large eddy structure (c.f. Townsend [15]). Therefore, we try an eddy viscosity and eddy diffusivity which are independent of radial position, that is,

$$\nu(\eta) = \text{constant} \equiv \frac{1}{R_T} \tag{16}$$

$$\alpha(\eta) = \text{constant} \equiv \frac{1}{P_T R_T} \tag{17}$$

where R_T and P_T will be referred to as the turbulent Reynolds' and Prandtl numbers respectively. Thus we have

$$\nu_e = \frac{1}{R_T} F_o^{1/3} x^{2/3} \tag{18}$$

and

$$\alpha_e = \frac{1}{P_T R_T} F_o^{1/3} x^{2/3} \tag{19}$$

Substitution into the equations of motion yields

$$-\frac{1}{3} f_1^2 - \frac{5}{3} \frac{f_1'}{\eta} \int_o^\eta \eta_1 f_1(\eta_1) d\eta_1 = \frac{1}{R_T} \frac{1}{\eta} \frac{d}{d\eta} (\eta f_1') + t_1 \tag{20}$$

$$-\frac{5}{3} f_1 t_1 - \frac{5}{3} \frac{t_1'}{\eta} \int_o^\eta \eta_1 f_1(\eta_1) d\eta_1 = \frac{1}{R_T P_T} \frac{1}{\eta} \frac{d}{d\eta} (\eta t_1') \tag{21}$$

These are ordinary differential equations for the two functions, f_1 and t_1 and can be solved directly when R_T, P_T and the boundary conditions are specified.

The appropriate boundary conditions are

$$f_1'(0) = 0, \quad f_1(\infty) = f_1'(\infty) = 0$$
$$t_1'(0) = 0, \quad t_1(\infty) = t_1'(\infty) = 0 \tag{22}$$

These simply state that the flow is symmetric about the axis and vanishes at infinity.

The solutions to these equations must satisfy some form of the integral constraint of equation (6). If we denote that fraction of the buoyancy which is carried by the mean flow as Q, the appropriate constraint in dimensionless form is

$$2\pi \int_o^\infty f_1(\eta)\, t_1(\eta) \eta d\eta = Q \tag{23}$$

The factor Q can take any value between zero and unity, the latter corresponding to a negligible turbulent contribution to the vertical heat transport. Note that this is the only place where the (unknown) turbulent contribution to the buoyancy integral enters the problem, and that the magnitude of this contribution in this formulation must be specified.

THE DEPENDENCE ON THE PARAMETERS R_T, P_T, AND Q.

We can illustrate the dependence of these equations on the coupled parameters by first carrying out the following transformation on equations (12) and (13).

$$\xi = \sqrt{R_T}\ \eta \tag{24}$$

$$f_1(\eta) \rightarrow f(\xi)\ ,\ t_1(\eta) \rightarrow t(\xi)\ ,\ k_1(\eta) \rightarrow R_T^{-\frac{1}{2}} k(\xi)$$

The governing equations can easily be shown to reduce to

$$-\frac{1}{3} f^2 - \frac{5}{3} \frac{f'}{\xi} \int_o^\xi \xi_1 f(\xi_1) d\xi_1 = \frac{1}{\xi} \frac{d}{d\xi} (\xi f') + t \tag{25}$$

$$-\frac{5}{3} ft - \frac{5}{3} \frac{t'}{\xi} \int_o^\xi \xi_1 f(\xi_1) d\xi_1 = \frac{1}{P_T} \frac{1}{\xi} \frac{d}{d\xi} (\xi t') \tag{26}$$

and the integral constraint is now given by

$$\int_o^\infty ft\ \xi d\xi = \frac{R_T Q}{2\pi} \tag{27}$$

In this form it is clear that the solutions to equations (25) and (26) depend only on the parameter P_T and are independent of R_T and Q. These latter dependencies enter only when the integral constraint of equation (27) is applied, and then only in the combination $R_T Q$. Thus an entire family of possible solutions is generated for each value of $R_T Q$. It is not until we map this solution back to physical coordinates by reversing the transformation of equations (24) that the actual dependence on R_T (or Q) enters. The choice of Q is, of course, limited by the physical constraints $0 < Q < 1$.

From the above it is clear that the magnitude and basic shape of the profiles are determined only by the product $R_T Q$. In particular, the centerline values are uniquely determined. From equation (24), it is also clear that the actual physical width (or spreading rate) is determined by R_T alone. If the centerline values are assumed determined by the data, then the profile width determines Q, the fraction of the total vertical heat (or buoyancy) transport due to the turbulent fluctuations. The reverse is, of course, true also.

THE EXACT SOLUTION OF YIH

Yih [7] was able to find exact solutions* satisfying the equations, boundary conditions, and integral constraints for the particular cases $P_T = 1.1$ and 2.0. The profiles were

$$f(\eta) = \frac{f_o}{[1+A\eta^2]^2} \tag{28}$$

$$t(\eta) = \frac{t_o}{[1+A\eta^2]^m} \tag{29}$$

where for $P_T = 1.1$, $m = 3$ and for $P_T = 2.0$, $m = 4$. As illustrated in the previous section the parameters f_o, t_o, A, P_T, R_T must be interrelated. These relationships are summarized in Table I.

TABLE I

P_T	1.1	2.0
m	3	4
$f_o^2/R_T Q$	$11/16\pi$	$25/36\pi$
$t_o/R_T Q$	$11/18\pi$	$125/240\pi$
f_o^2/t_o	9/8	3/4
$Af_o/t_o R_T$	10/64	11/64
$A/R_T^{3/2} Q^{1/2}$	0.065	0.057

* We present modified forms of Yih's solutions to include the Q-factor.

Yih's solutions have been presented along with the numerical results in Figures (2)-(4).

NUMERICAL SOLUTION FOR ARBITRARY PRANDTL NUMBER

The coupled nonlinear second order ordinary differential equations given by equations (25) and (26) were solved numerically to give $f_1(\xi)$ and $t_1(\xi)$. The numerical solution involved an iterative process in which at each iteration step the solutions to linearized versions of (25) and (26) were found. The linearization of equation (25) was accomplished by specifying that $f(\xi)$ and $\int_0^\xi \xi_1 f(\xi_1)d\xi_1$ are known and given by the previous iteration. The linearized equations were central differenced yielding a system of algebraic equations with a tridiagonal matrix whose solution was easily accomplished by an elimination process.

The solution was initiated using simple geometric functions to approximate $t_1(\eta)$ and $f_1(\eta)$ over the η range of interest. The program iterates upon the functions $t_1(\eta)$ and $f_1(\eta)$ until convergence is attained. The solution provides the functional relations $t_1(\eta)$ and $f_1(\eta)$ for a given R_T and P_T, and computes R_TQ, the value of the mean buoyancy integral.

Two hundred grid points were found to provide an increment in η which insured accuracy with reasonably fast convergence. The boundary conditions at infinity were known from the physics of the problem (equation 22) and experimental data were used to bound values at $f_1(o)$ and $t_1(o)$. Infinity was established by extending the solution in the radial direction to a value at which the buoyancy integral was constant. The profiles calculated by this technique were identical to those obtained at considerably greater expense and for a more limited range of R_T and P_T by Hamilton and George [8] using a shooting method.

RESULTS OF THE CALCULATIONS AND COMPARISON WITH EXPERIMENT

It was shown earlier that the form of the solutions depended only on the variables P_T and R_TQ. Figure (2) shows the interdependence of the centerline velocity and temperature as functions of the parameters R_TQ and P_T. The calculations cover the range of the existing data ($9 \le t_1(o) \le 13.7$, $2.7 \le f_1(o) \le 4.7$). Also shown are the lines corresponding to Yih's closed form solutions for $P_T = 1.1$ and 2, and the aforementioned calculations of Hamilton and George [8].

The experimental values obtained by Schmidt [9], Yih [10] and George, Alpert, and Tamanini [5] are also shown. The measurements of Schmidt have long been suspect since they do not satisfy the momentum equation (c.f. [10]). The measurements of Yih have recently been questioned by Yih himself [7] because of problems with the velocity probe, and by George et al. [5] who questioned whether Yih's flow development length was long enough to allow sufficient momentum buildup from a heat source to achieve an asymptotic state. In view of the fact that turbulent Prandtl numbers are generally accepted to be near unity for free turbulent shear flows, figure (2) indicates that these suspicions about the earliest measurements are probably well-founded, and they will not be used further here.

Figures (3) and (4) show the calculated profiles of velocity and temperature for P_T=1 and several values of R_TQ. Also shown for comparison is the data of reference [5]. The agreement between the measured profiles and those calculated for $R_TQ = 50$ is striking.

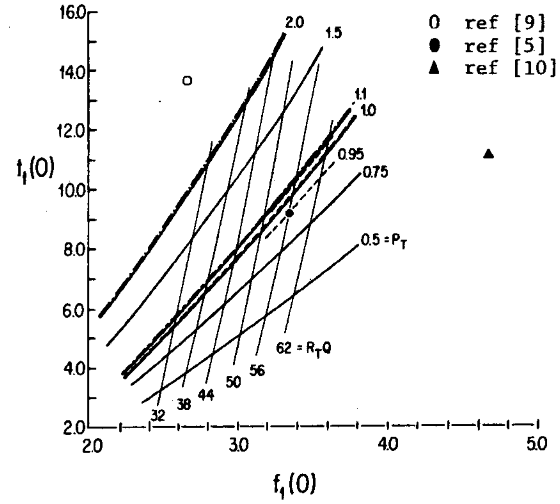

Figure 2 - Plot showing interrelation of centerline values of f_1 and t_1 as functions of parameters R_TQ and P_T. Solid lines show results of present study. Results of ref [8] shown for P_T = 0.95, 1.0, and 1.1 with dashed lines. Analytical results of ref [7] coincide with present study for P_T = 1.1, 2.0 (eqns. 28 and 29).

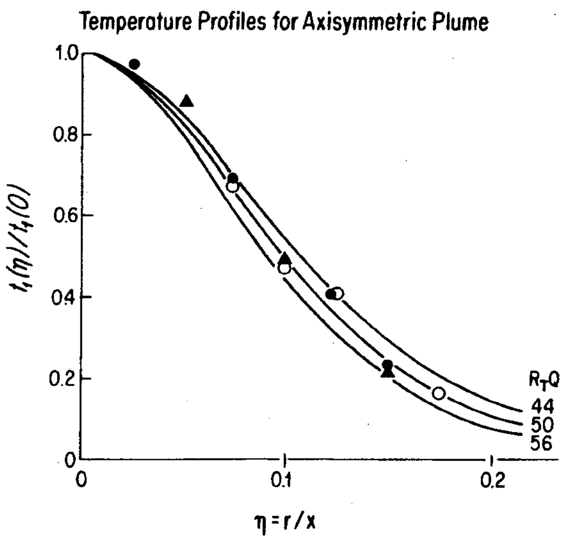

Figure 3 - Temperature profiles (normalized to centerline value) for P_T = 1.0, R_TQ = 44, 50, 56 with experimental data of ref [5].

As might be expected from the results shown in figure (2), Yih's solution for $P_T = 1.1$ is very close to those shown above. In fact when the width parameter is chosen to be A = 26, the profiles given by equations (28) and (29), when normalized by their centerline values, are indistinguishable from the $R_TQ = 50$ contours in figures (3) and (4). In view of this, Yih's profiles should be used in place of the commonly used Gaussian profiles whenever an analytical expression is desired since the fit to the velocity profile is far superior. (c.f. reference [5]).

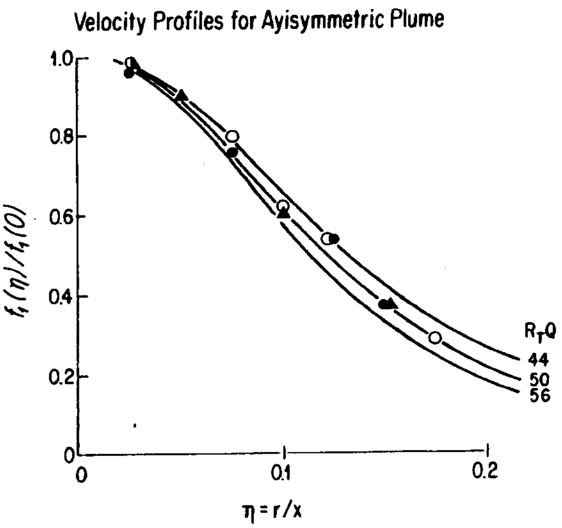

Figure 4 - Velocity profiles (normalized to centerline value) for $P_T = 1.0$, $R_TQ = 44, 50, 56$ with experimental data of ref. [5].

Figures (5) and (6) show the calculated Reynolds stress and radial turbulent heat flux corresponding to the $R_TQ = 50$, $P_T = 1.0$ case presented above. In order to obtain these plots it was necessary to know R_T explicitly, thereby necessitating a choice for Q. We have selected Q = 0.85 which corresponds to the estimate of George et al. [5] based on measurements of $\overline{u\theta}$. Also plotted are the Reynolds stress and radial turbulent heat flux measurements of Beuther and George [11]. Agreement between the calculated and measured values is excellent near the centerline and in the core region of the plume. The fact that the calculated and measured values of the heat flux deviate at large η may be due to the non-negligible influence of the vertical turbulent heat flux on the temperature equation in this region (see discussion below). The agreement between calculated and measured Reynolds stress at <u>all</u> values of η is surprising and gratifying.

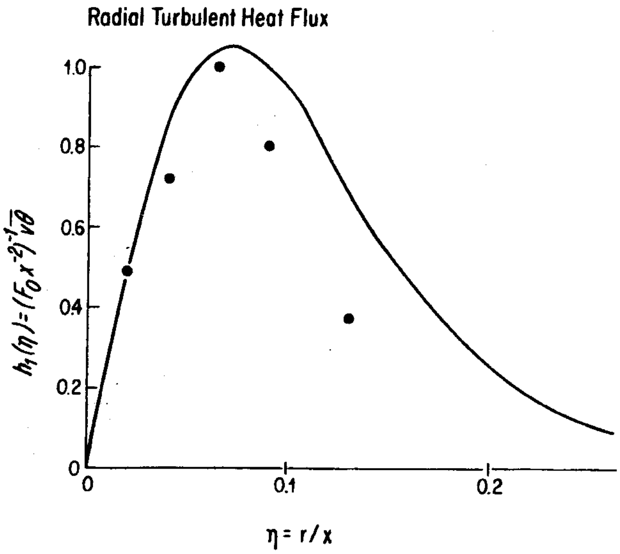

Figure 5 - Calculated radial heat flux for $P_T = 1.0$, $R_TQ = 50$, Q = 0.85 with experimental data of ref. [11].

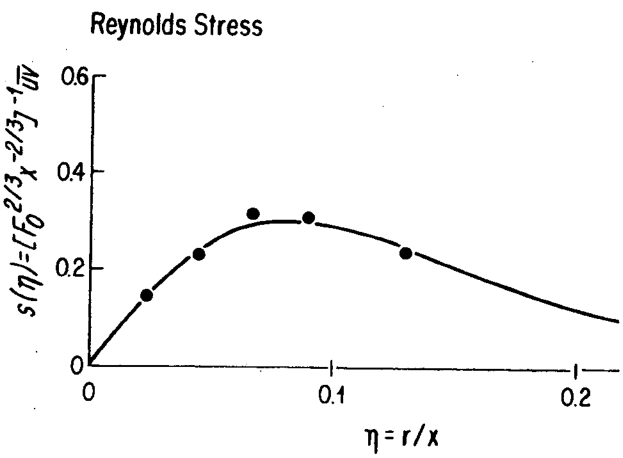

Figure 6 - Calculated Reynolds stress for $P_T = 1.0$, $R_TQ = 50$, Q = 0.85 with experimental data of ref [11].

DISCUSSION

It is both surprising and somewhat misleading that such excellent agreement can be obtained between measured profiles and the predictions of a simple eddy viscosity model which accounts for the large turbulence contribution to the vertical heat flux in only the most elementary manner. It is interesting to speculate on the reasons for this success. This is particularly important when considering extended applications of this model to other problems dominated by buoyancy.

The vertical turbulent heat flux profiles ($\overline{u\theta}$) measured by George et al. [5] are shown in figure (7). It is obvious that this profile is considerably broader than the temperature or velocity profiles (normalized to the same maximum value). In fact, over the core region of the plume, it is reasonable to approximate $\overline{u\theta}$ by a simple top hat function. This constancy in the core region accounts for the fact that it has no influence over the shape of the velocity and temperature profiles except through integral parameter Q. As the intermittency at the outer edge begins to play a role in the actual profile shape. It makes sense that this effect would first be seen in the temperature equation, and probably accounts for the deviations between calculated and measured values of radial heat flux.

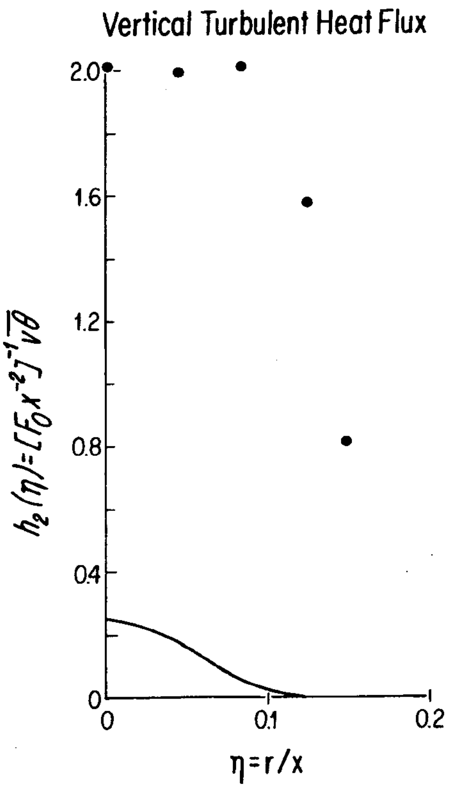

Figure 7 - Calculated vertical heat flux for $P_T = 1.0$, $R_T = 60$ with experimental data of ref. [5].

There is reason to believe that this relative constancy of the vertical turbulent heat flux is exhibited in, at least, two-dimensional plumes and the main part of the natural convection turbulent boundary layer next to a heated vertical flat plate (c.f. ref. [12] and [13]). Calculations have already been carried out by the present authors for the two-dimensional plume but are not presented here because of the inadequate experimental data base. (Q has not been measured). Efforts are currently being made to compute the profiles for the main part of the natural convection boundary layer using a constant eddy viscosity and matching the profiles to the buoyant sublayer profiles of George and Capp [14] as the wall is approached.

Finally, an attempt was made to calculate directly the vertical turbulent heat flux, $-\overline{u\theta}$ using the same value for the eddy diffusivity as used for the radial component. This corresponds to the commonly assumed isotropic medium model

$$-\overline{u_i\theta} = \alpha_e \frac{\partial T}{\partial x_i}$$

The result for the case $P_T = 1$, $R_T = 60$ is plotted in figure (7). It is clear that the actual vertical heat flux is substantially underestimated.

The reasons for this are obvious when one considers the dynamical equation for $-\overline{u\theta}$. Unlike the equation for $\overline{v\theta}$ where the dominate source terms arise only from mean gradient terms, the $\overline{u\theta}$ equation has, in addition, a direct buoyancy source term which depends only on gravity and the temperature fluctuations. Thus it is not surprising that a "gradient transport" model fails in accounting for its behavior. One can infer from this that isotropic eddy viscosity models will probably always fail in problems involving buoyancy when the directions of the mean flow and gravity are aligned. Note that this does not exclude the possibility that additional tricks (like the Q factor introduced above) can be employed to account independently for the vertical turbulent heat flux contribution to the problem.

SUMMARY AND CONCLUSIONS

It has been shown that a simple eddy viscosity model can accurately predict the velocity and temperature profiles of a simple buoyant plume in a neutral environment, if the contribution of the turbulence to the vertical heat transport is accounted for separately. This result was anticipated from the fact that the flow could be characterized by a single time and a single length scale.

The computed centerline values proved to be useful in sorting conflicting experimental data; it was concluded that turbulent Prandtl numbers near unity gave the best results. The best fit to the experimental data of references [5] and [11] was given by $P_T = 1.0$ and

$$\nu_e = \frac{1}{60} F_o^{1/3} x^{2/3}$$

This corresponds to $R_T Q = 50$ for which 85% of the vertical heat transport is carried by the mean flow.

Since the computed profiles for turbulent Prandtl numbers near unity were virtually indistinguishable from Yih's analytical solution for $P_T = 1.1$ (eqns. 28 and 29), it is strongly recommended that Yih's solution be used as empirical profiles (with constants to be determined from the data) in place of the oft-used Gaussian forms. We recommend

$$f(\eta) = \frac{3.4}{[1+26\eta^2]^2}$$

and

$$t(\eta) = \frac{9.1}{[1+26\eta^2]^3}$$

for the velocity and temperature respectively.

It is interesting to speculate as to whether the method used here can be extended to more complicated problems such as forced plumes and stratified environments. It is easy to show that either of these situations introduces new time and length scales into the problem. When the ratio of these new scales to the old is small, it is reasonable to expect the eddy viscosity will be nearly independent of it. Moreover, an expansion in small values of the length and time scale ratio might prove useful. It is not immediately obvious how to account for the variation of the vertical turbulent heat transport in such a model.

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