

An Introduction to Natural Convection Flows

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Chapter 1

Introduction

1.1 Some examples

Natural convection flows truly are all around us. Any time there is something to create density difference other than the flow itself, there is the potential for a natural convection flow. For example, the heating of air by the sun can cause local parcels of air to be lighter than their environment. The result is that they rise. If there is moisture in the air, the condensation that results from the adiabatic cooling as the air rises can cause further heat release and an even faster rate of rise. Soaring birds use these thermals to circle while they hunt, or maybe just for fun. Thunderstorms and hurricanes – which can be a lot less fun – are extreme manifestations of the same phenomenon. One of my own personal favorites was Hurricane Barry in 1995, which blew me off my intended path from America to Europe.¹

The opposite happens if air is cooled locally. It falls. This can be as simple as causing down drafts next to the walls of your house on a cold day, to the winds that can suddenly rush down mountainside (the *bora* or *mistral* is an example). They result from the overflow of air inside a mountain ridge that has been cooled by radiation. I nearly lost a boat in Croatia by not being aware of this phenomenon.

But natural convection flows can be easily be observed on any stove top. Simply heat a viscous fluid from below gently and the famous Bénard hexagonal convection cells result. Try it yourself sometime: just a thin layer of tomato soup and a bit of milk for visualization. The same principle is used to for emergency cooling of nuclear reactors like Super Phénix, and many others as well.

Such flows also play an important role in cooling applications. All you need to see them is light passing through at an angle so they make patterns on the wall. The humble incandescent light bulb is one example. Your cellphone is another. The most exotic laptops do not have an internal fan, depending on natural convection through ‘heat pipes’ to carry the heat away.

Natural convection flows are the heart of our attempts to build healthy build-

¹Instead of sailing a great circle from Nova Scotia to Scotland, we ended up in the Azores.:-)

ings and save energy. Architects for centuries have understood the importance of using the tendency of warm air to rise and cool air fall. Modern architects have taken this to another level – the office building across from the British parliament is but one excellent example.

Even the cooling flow passages inside of rapidly rotating gas turbine engines can behave as a natural convection flow. Here it is the rapid rotation of flow and the resulting centripetal accelerations which providing the gravity. The density differences of course result from the hot flow passing on the other side of the blades and the cooling flow which has by-passed the combustor. Interestingly, even the Taylor column phenomenon of stratified flows over mountains and hills has its counterpart inside the cooling passages, with the result that the most of flow passage can be effectively blocked.

1.2 The really big problem is turbulence

For the fluid dynamicists, there are few problems more interesting than natural convection flows, if for no other reason than that they can be observed almost everywhere. There are lots of ways to analyze laminar flows, and laminar flow analysis has a rich and well-documented history. But very few flows are laminar. And those that start laminar, don't stay that way for very long. A simple smoldering cigarette exhibits in a decimeter or less all of the complications that defy analysis and frustrate engineers. Whenever I put the cold cream into my coffee I marvel at the complex events which ensue. First the cold cream seems to completely disappear, as it drops to the bottom of the cup. Then it reappears, usually rising near the center. And when it reaches the top, it starts its passage back to the bottom. One cycle is all it takes before the real problem of turbulence becomes clear. By the second time around the flow has usually exploded into the complex phenomenon we call turbulence. It is easy to see from even this simple experiment why natural convection flows played such an important role in the early development of chaos theory.

Turbulence both fascinates and frustrates. Almost no one can fail to be impressed and awed by the myriad patterns and forms that appear before us constantly. Natural convection flows especially like being turbulent; the non-barotropic term in the vorticity equation rapidly augmenting the stretching and turning term that dominates aerodynamics flows. But the same term can also dampen or even squash turbulence entirely. To see this simply go outside on a cold clear night and notice how the wind can entirely disappear, even when the jet stream is not far above. Buoyancy confounds our usual turbulence models and makes reliable turbulence prediction almost impossible. If you have any doubts, count the number of times you have planned a summer picnic and been assured of fine weather, only to have it ruined by a thunder storm. Or how many times you have canceled an excursion due to an inclement weather forecast, only to later discover to your total chagrin that the weather was ideal. Meteorologists try to

soothe us with probabilities, but the truth is that most of the time they really have no idea where or if a thunderstorm will really occur.

1.3 The really big person

By far the most important person in the history of our studies of natural convection was Joseph Valentin Boussinesq. He was French, of course, and from 1872 to 1886 he was professor at the Faculty of Sciences at Lille. There is an excellent summary and picture in wikipedia. And if you ever walk in the lab at LML which houses the 20 meter long wind tunnel, stop to read the plaque about him that hangs near the entrance.

The first big idea (at least for the turbulence community) is still used by turbulence modelers today. It was that of the eddy viscosity (or turbulent 'viscosity'), which relates the flux of a quantity to its gradient. The second big idea is often called the Boussinesq approximation, and it forms the basis of almost all theoretical studies of natural convection. We shall discuss it in detail at the end of the next chapter, and use it repeatedly.

1.4 What can we learn in three hours?

This is primarily a course in numerical analysis. I am an experimentalist and a theoretician. My mission is not to talk about numerical techniques at all. But instead to talk about the underlying equations as they affect natural convection flows. The reason I accepted this responsibility is in part because I like coming to France, and in part because I very happy for the opportunity to talk about something I like and think is very important.

But there is another darker reason why I have agreed to give these lectures. When I started working in natural convection in the 1970's, I really thought we would be much farther ahead in our understanding and in our ability to predict them numerically. The fact is that in my opinion we have made almost no real progress in the past 30 years. And the reason I believe is that almost all of the funding and effort in natural convection research has gone into attempts to do experiments and numerical analysis (DNS, RANS, LES) that could have been recognized at the outset to simply not measure up. Some of these efforts have been truly heroic in terms of the effort and ingenuity needed to perform them. But the reason they contributed little is that they began with a very poor understanding of the problem. And in particular, they failed to recognize what was necessary to truly make a contribution to our understanding – as opposed to simply using a new experimental technique or proving that a bigger computation could be done.

I hope through these three hours of lecture (especially the last) to show that these efforts (especially the numerical ones) were doomed from the outset: there simply was not enough computational power to compute an interesting flow. Or the experiments were far far too small. I'm pretty sure the same is true today.

But you are just beginning your careers as fluid mechanics and cfd'ers. You don't have to fall into these traps. Sometimes the best experiment or simulation is the one that was NOT done. The people who fund our work really don't like paying for something twice. So if you do something that really was not quite ready to be done, you preclude someone (often yourself) from doing it correctly.

But unless I miss my guess, most of you are young enough that long before you are my age, the problems (especially the natural convection problems) that have so long frustrated us may be possible to solve numerically. But you will only recognize when that moment has come if you know and understand what the problem is. Helping you to recognize the real problems is the primary goal of these lectures.

In the following pages are the beginnings of some lectures about natural convection. In chapter two we shall review the equations of fluid mechanics and talk about how we use them to understand natural convection. Chapter three will take us into a detailed discussion of the thermal energy equation. One might think these would all be well-known and canned stuff to learn from a book, but my detailed discussions with Poul S. Larsen of the Danish Technical University have convinced me that few really understand, and most believe things that are incorrect or unjustifiable. The remaining chapters are not really chapters at all, but talking points if you will. I have picked examples from my own work, to illustrate the main points. But these chapters (and the associated papers) contain perhaps the most important points to be made in these lectures – they present the state of our knowledge of turbulence scales as they affect natural convection flows: length scales, integral scales, microscales, inner and outer scales. The most important single point is that unless a simulation or experiment can achieve at least a minimum value of the ratios of these scales, it is probably a waste of time and money. Unfortunately for natural convection flows the numbers required to be interesting are huge – for example a Reynolds number of 10^{12} would correspond to the highest Reynolds number flows on earth. But a Rayleigh or Grashof number of 10^{12} is only just at the margin of beginning to be interesting.

I would love to have provided detailed references, but frankly there are none that I really like, and all will somewhat mislead. (Feel free to suggest them to me if you have found some. I most certainly am not very familiar with the French literature.) Therefore the focus will be on reason, and encouraging you to think for yourself. But this should be no problem for you. The French have a long and rich history of advocating reason, an academic heritage if you will of which I am proud to claim I am a part.

Chapter 2

Momentum and Mass Conservation

The goal of this chapter is to lay out the basic equations (as best we know them) on which we will later make approximations. Since the approximations necessary for theoretical analysis sometimes differ significantly from those used in numerical analysis, it is quite important to have a firm grasp on exactly what the underlying equations are. This is especially important when comparing either to experiments, since nature does not know about our approximations. :-)

2.1 The momentum equation

Any study of any flow rightly begins with the equations of motion. We shall only be concerned with their spatial (or Eulerian) formulation, at least in these lectures. We can summarize them as follows:

$$\tilde{\rho} \frac{D\tilde{u}_i}{Dt} = \frac{\partial \tilde{\tau}_{ji}}{\partial x_j} + \tilde{\rho} f_i \quad (2.1)$$

where we have used the Einstein summation convention (repeated index means an implied summation over 1,2 and 3.) We use a tilde $\tilde{}$ to distinguish instantaneous quantities from the averaged and fluctuating quantities we will use later. The symbol $\tilde{\rho}$ stands for the instantaneous density, \tilde{u}_i represents the i -th component of the velocity vector, $\tilde{\tau}_{ij}$ is the instantaneous stress tensor, and f_i is the body force per unit mass. The body force of primary interest for us will be gravity, but centripetal accelerations can be important as well (as noted in the introduction).

D/Dt stands for the material(or Stokes) derivative defined by:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \quad (2.2)$$

It is really the time rate-of-change that would be seen if riding on a fluid particle which at that exact moment, time t , is passing through the spatial point, \vec{x} . Thus

the left-hand-side of equation 2.1 is really just the representation at the point, x , and time, t , of the acceleration of the fluid (particle) which happens to be at that point. Therefore equation 2.1 is just the Eulerian expression of Newton's Law for an accelerating mass acted upon by contact and body forces.

2.2 The Stress Tensor

The stress tensor can be used to reconstruct the contact force per unit area, $t^{(n)}$, exerted by the fluid on one side of a fluid surface on the other side. Using Cauchy's stress principle:

$$\tilde{t}_i^{(n)}(\vec{x}, t) = \tilde{\tau}_{ji}(\vec{x}, t)n_j(\vec{x}, t) \quad (2.3)$$

where $n_i(\vec{x}, t)$ is the unit outward normal to the surface which can be in motion. Most of the time we shall not write out explicitly the dependence of any of the relations on space and time, but it is always implied.

In a fluid without stress couples, conservation of angular momentum dictates that $\tilde{\tau}_{ji}$, be a symmetric tensor; i.e., $\tilde{\tau}_{ji} = \tilde{\tau}_{ij}$. We shall always assume this to be the case.

2.2.1 Mechanical pressure

It is possible to define a 'mechanical pressure' as the negative of the mean normal stress; i.e.,

$$\tilde{p}(\vec{x}, t) = -\frac{1}{3}\tilde{\tau}_{kk}(\vec{x}, t) \quad (2.4)$$

The reason for the negative sign is that we intuitively think of pressure as acting inward. It is easy to see from Cauchy's stress principle above that we have accomplished this since the pressure acts in a direction opposite to the outward normal.

There is, of course, no reason to believe that the mechanical pressure is the same as the thermodynamic pressure of equilibrium thermodynamics. Establishing their relation is quite important, however, since we will need to employ thermodynamic relations to close our equations. In fact, for a fluid at rest they can be shown to be the same (see ??). For a fluid that is moving this is more problematical and will be addressed below. But for all purposes in these notes the mechanical and thermodynamic pressures can be assumed to be equal.

2.2.2 Viscous stress

The mechanical pressure can be used to subtract an *isotropic* part from the stress tensor, leaving only the deviatoric (or viscous) part, $\tilde{\tau}_{ji}^{(v)}$ as:

$$\tilde{\tau}_{ji}^{(v)} = \tilde{\tau}_{ji} - \frac{1}{3}\tilde{\tau}_{kk}\delta_{ij} \quad (2.5)$$

$$= \tilde{\tau}_{ji} - (-p\delta_{ji}) \quad (2.6)$$

where δ_{ji} is the Kronecker delta or isotropic tensor. It follows immediately that the complete stress tensor can be then written as:

$$\tilde{\tau}_{ji} = -p\delta_{ij} + \tilde{\tau}_{ji}^{(v)} \quad (2.7)$$

This can be substituted directly into equation 2.1 to yield the most common form of the momentum equation as:

$$\tilde{\rho} \frac{D\tilde{u}_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tilde{\tau}_{ji}^{(v)}}{\partial x_j} + \tilde{\rho} f_i \quad (2.8)$$

2.2.3 Newtonian fluid

By a series of very restrictive assumptions the constitutive equation for a viscous fluid without memory and which responds only linearly to its immediate neighborhood can be written as:

$$\tilde{\tau}_{ji}^{(v)} = 2\mu[s_{ji} - \frac{1}{3}s_{kk}\delta_{ji}] \quad (2.9)$$

where s_{ji} is the strain rate tensor defined by:

$$s_{ji} = \frac{1}{2} \left[\frac{\partial \tilde{u}_j}{\partial x_i} + \frac{\partial \tilde{u}_i}{\partial x_j} \right] \quad (2.10)$$

A fluid that behaves this way is called a **Newtonian fluid**. Few fluids actually do; but fortunately for us both water and air (and most gases) do under all but the most extreme conditions. A number of common liquids do as well, most with rather simple chemical composition. The quantity, μ , is the *viscosity* of the fluid, which should not be confused with the **kinematic viscosity**, ν which is the viscosity normalized by the density; i.e., $\nu = \mu/\tilde{\rho}$. There can also be included a second viscosity term in equation 2.9 which is also proportional to the expansion, s_{kk} , but it is almost always neglected unless the strain rates are very high (e.g., intense ultrasound).

In general, μ is a function of the temperature; and even more so ν . This is a source of considerable added complexity in many buoyancy-dominated flows, since the density variations are consequence of temperature variations within the fluid. It can render analysis difficult to impossible. It is less of a difficulty for numerical analysis of at least laminar flow, but even for these it can complicate the interpretation of data. For turbulent flows it becomes an almost impossible mess.

The Newtonian constitutive equation (equation 2.9) can be substituted into the momentum equation, (equation 2.8) to obtain the form of the momentum equation which will receive the most attention in this course as:

$$\tilde{\rho} \frac{D\tilde{u}_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left\{ \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \right] \right\} + \tilde{\rho} f_i \quad (2.11)$$

Because of the temperature (and hence spatial dependence) of the viscosity, μ , this equation cannot be further reduced to the simple form of the Navier-Stokes equations used for most aerodynamic flows, even when incompressible. This is a source of great difficulty in many natural convection flows which are driven by thermal variations, and it especially complicates the development of scaling laws, theories, etc. Numerical analysis (laminar or DNS) can, of course, account for them since a thermal computation must be performed in parallel. But for turbulence closure, the fluctuations in viscosity (and in particular their correlations with other variables) must, in principle at least, be treated like any other turbulent quantity. The difficulties presented by this cause nightmares for serious scholars who worry about combustion processes, for example.

2.3 Mass Conservation

The equations of mass conservation can be found in any fluids text. In Eulerian form they become:

$$\frac{D\tilde{\rho}}{Dt} + \tilde{\rho} \frac{\partial \tilde{u}_k}{\partial x_k} = \sigma \quad (2.12)$$

where $\sigma(\vec{x}, t)$ represents any sources of mass per unit volume. Unless the buoyancy is due to the presence of air bubbles, most natural convection flows are source free, so the so-called **continuity equation** reduces to:

$$\frac{D\tilde{\rho}}{Dt} + \tilde{\rho} \frac{\partial \tilde{u}_k}{\partial x_k} = 0 \quad (2.13)$$

or

$$\frac{1}{\tilde{\rho}} \frac{D\tilde{\rho}}{Dt} = -\frac{\partial \tilde{u}_k}{\partial x_k} \quad (2.14)$$

The term on the left is the rate of change of density per unit density following a fluid material point. Since the whole idea of a fluid particle is that its mass is fixed, the continuity equation is basically saying that changes in density must be compensated for by a change in volume. But the term on the right can easily be shown to be directly proportional to the rate of change of volume (per unit mass) per unit volume (per unit mass), exactly as required.

In fact, **most natural convection flows can be approximated as incompressible** since their Mach number's are usually quite low, which implies that either:

$$\frac{D\tilde{\rho}}{Dt} = 0 \quad (2.15)$$

or

$$\frac{\partial \tilde{u}_k}{\partial x_k} = 0 \quad (2.16)$$

It is extremely important to note that incompressibility does **NOT** require the density be constant throughout the field. In fact we shall see that the whole idea of a natural convection flow is that the density varies throughout the field enough for gravity to differentially affect it. The fact that the density varies throughout the field is not at all incompatible with incompressibility, however, since incompressibility only requires that density of a particular fluid particle remain constant. And that is exactly the meaning of $D\tilde{\rho}/Dt = 0$.

2.4 A common mistake, especially by CFD'ers

An alternative version of the momentum equation above can be derived by expanding the material derivatives of eqns 2.1 and 2.13, multiplying the latter by \tilde{u}_i and adding it to the first, then rearranging the terms it is possible to obtain:

$$\frac{\partial}{\partial t} \tilde{\rho} \tilde{u}_i + \frac{\partial}{\partial x_j} \tilde{\rho} \tilde{u}_i \tilde{u}_j = \frac{\partial \tilde{\tau}_{ji}}{\partial x_j} + \tilde{\rho} f_i \quad (2.17)$$

This is the so-called ‘‘conservative form’’ of the momentum equation, which has some advantages for numerical computations.

A similar rearrangement can be performed on the continuity equation with the result:

$$\frac{\partial}{\partial t} \tilde{\rho} + \frac{\partial}{\partial x_k} \tilde{\rho} \tilde{u}_k = 0 \quad (2.18)$$

When used together, these two equations together contain exactly the same information as the equations from which they were derived.

So what then, you are probably asking, is the mistake? The mistake is in believing (and sometimes teaching) that the other forms of the momentum equation (e.g., equation 2.1) are only valid for incompressible or even constant density flows. This is, of course, completely false, as you can easily prove for yourself by simply reversing the procedure above. Part of the reason for the confusion lies, I am sure, in the fact that the density is outside the material derivative in equation 2.1; but that is a natural consequence of the Reynolds transport theorem and has nothing to do with incompressibility.

2.5 Pressure: the complexity of our problem

Recently I and my co-workers (??) while playing with the equations for sound propagation rearranged the momentum and continuity equations yet another way.

Without making assumptions we took the divergence of the equation 2.8 and used the continuity equation to obtain:

$$\frac{D}{Dt} \left[\frac{1}{\tilde{\rho}} \frac{D\tilde{\rho}}{Dt} \right] - \frac{\partial}{\partial x_j} \left[\frac{1}{\tilde{\rho}} \frac{\partial \tilde{p}}{\partial x_j} \right] = \frac{\partial \tilde{u}_i}{\partial x_j} \frac{\partial \tilde{u}_j}{\partial x_i} + \left\{ \frac{\partial^2 \tilde{\tau}_{ij}^{(v)}}{\partial x_i \partial x_j} + \tilde{\rho} \frac{\partial f_i}{\partial x_i} - \frac{D\sigma}{Dt} \right\} \quad (2.19)$$

If we restrict ourselves to a flow with no mass sources ($\sigma = 0$) and no body force gradients ($\partial f_i / \partial x_i = 0$) and neglect the viscous stress, this reduces to:

$$\frac{D}{Dt} \left[\frac{1}{\tilde{\rho}} \frac{D\tilde{\rho}}{Dt} \right] - \frac{\partial}{\partial x_j} \left[\frac{1}{\tilde{\rho}} \frac{\partial \tilde{p}}{\partial x_j} \right] = \frac{\partial \tilde{u}_i}{\partial x_j} \frac{\partial \tilde{u}_j}{\partial x_i} = \tilde{s}_{ij} \tilde{s}_{ij} - \tilde{\Omega}_{ij} \tilde{\Omega}_{ij} \quad (2.20)$$

On the right-hand side we have decomposed the velocity gradient tensor into its symmetric part and antisymmetric parts:

$$\frac{\partial \tilde{u}_i}{\partial x_j} = \tilde{s}_{ij} + \tilde{\Omega}_{ij} \quad (2.21)$$

where the antisymmetric part, the rotation rate tensor $\tilde{\Omega}_{ij}$, is defined by:

$$\omega_{ij} = \frac{1}{2} \left[\frac{\partial \tilde{u}_i}{\partial x_j} - \frac{\partial \tilde{u}_j}{\partial x_i} \right] \quad (2.22)$$

Note that $\tilde{\Omega}$ can be determined only from the vorticity, $\omega_i = \epsilon_{ijk} \partial \tilde{u}_j / \partial x_k$; i.e.

$$\tilde{\Omega}_{ij} = -\frac{1}{2} \epsilon_{ijk} \omega_k \quad (2.23)$$

2.5.1 Constant density case

Now from equation 2.20 a couple of things become clear. First note what happens if the density is the same everywhere (i.e., $\tilde{\rho} = \rho_o = \text{constant}$). Then the pressure, and especially the pressure fluctuations, are entirely determined by the imbalance between the squares of the strain-rate and rotation rate tensors. The left-hand side in fact just becomes the Laplacian of the pressure, since the density is constant and can be factored out; i.e.,

$$-\frac{1}{\rho_o} \nabla^2 \tilde{p} = \tilde{s}_{ij} \tilde{s}_{ij} - \tilde{\Omega}_{ij} \tilde{\Omega}_{ij} \quad (2.24)$$

The solutions to this equation are well-understood and usually involve Green's functions. The most important property (from our perspective at least) is their global nature. For example, in a three-dimensional infinite domain the solution for the pressure field is given by:

$$\frac{1}{\rho_o} \tilde{p}(\vec{x}, t) = -\frac{1}{4\pi} \int \int \int_{\text{allspace}} \left[\frac{\partial \tilde{u}_i}{\partial y_j}(\vec{y}, t) \frac{\partial \tilde{u}_j}{\partial y_i}(\vec{y}, t) \right] \frac{d\vec{y}}{|\vec{y} - \vec{x}|} \quad (2.25)$$

Clearly the solution at any location, say \vec{x} , depends on the weighted integral of the squared velocity gradient tensor at all other points, \vec{y} , at the same instant in time, t . In general, this presents a difficult problem for any numerical scheme because of the numerical problems it presents and the sensitivity to instantaneous boundary conditions. This is, in fact, the reason many codes are designed to be ‘weakly compressible’. The non-local nature of the pressure field presents an almost impossible problem for the turbulence closure modeler, since all models are based on ‘local’ approximations (e.g., gradient transport, return-to-isotropy). (In fact some models try to account for this at least a bit; e.g., the v2-f model) But laboratory experiments (at least subsonic ones) are not immune either, since they too must be performed inside enclosures, so the entire pressure field at any location knows what tricks we are trying to perform at the boundaries.

2.5.2 Incompressible case

Most of the natural convection flows that are of interest can be assumed to be nearly incompressible. At first glance it might seem that the incompressible form of equation 2.20 is the same as the constant density case, but this is an illusion. The presence of the density under the derivative in the pressure makes an important difference which can be seen by expanding it out to obtain:

$$\nabla^2 \tilde{p} = -\tilde{\rho} \frac{\partial \tilde{u}_i}{\partial x_j} \frac{\partial \tilde{u}_j}{\partial x_i} + \frac{1}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial x_i} \frac{\partial \tilde{p}}{\partial x_i} \quad (2.26)$$

where we have placed the extra term on the right-hand side. Clearly, *even in an incompressible flow the density and pressure fields are coupled*. We can see the importance of this another way if we use Euler’s equation (since we have neglected viscous terms already) to substitute for the pressure gradient on the right-hand side to obtain:

$$\nabla^2 \tilde{p} = -\tilde{\rho} \frac{\partial \tilde{u}_i}{\partial x_j} \frac{\partial \tilde{u}_j}{\partial x_i} - \frac{\partial \tilde{\rho}}{\partial x_i} \frac{D \tilde{u}_i}{Dt} \quad (2.27)$$

The presence of the time derivative on the right-hand-side makes it clear that now there is the possibility of action at one point in space affecting the flow at other places at a later time. This is the source, in fact, of the internal wave motions which have quite fascinating properties in their own right. (For example, the energy can go in one direction, while the phase moves in another.) And this considerably complicates both numerical analysis and experiments in flows with density stratification, not least because the boundary conditions imposed must find a way to either damp the waves at the boundary or let the energy out.

2.6 Gravity

The most common body force of interest in natural convection is that of gravity. (As noted earlier natural convection-like phenomena can also occur in rotating

machinery.) The gravitational force, of course, varies inversely with distance from the gravitating body (unless you are in it). Even when the entire atmosphere is being considered, however, it is almost always possible to forget about self-gravitating effects, and even the variation of gravity with height, and treat it as a simple constant vector. This, of course, does not apply to the analysis of motions inside the earth's core, for example, where neither of these approximations would be appropriate.

Life is usually the simplest when one of the primary flow directions is chosen to correspond to the direction of the gravitation vector. Exactly which direction is chosen varies from field to field. Meteorologists seem to always choose x_3 as positive upward so $\vec{g} = (0, 0, -g)$. An oceanographer might do the opposite and place the origin in the surface of the ocean with positive x_3 pointing downwards so $\vec{g} = (0, 0, +g)$. Engineers, on the other hand usually like the principal flow direction to be x_1 , so the gravitational vector can show up anywhere, depending on which direction relative to it the flow is pointing. For example in the plume and vertical wall examples we shall consider later, we shall use $\vec{g} = (-g, 0, 0)$, since the flow will be considered positive upward. While this is a bit annoying if you like things to be consistent, just consider it the price you pay for working in a field in which so many disciplines have an interest.

For now we shall simply avoid the problem entirely and take $f_i = g_i$. Thus our momentum equation, including gravity explicitly, can be written as:

$$\tilde{\rho} \frac{D\tilde{u}_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tilde{\tau}_{ji}^{(v)}}{\partial x_j} + \tilde{\rho} g_i \quad (2.28)$$

2.7 The Boussinesq approximation

As noted earlier, most of the flows we encountered can be considered incompressible. But this only solves part of our problem, since the density at each point can still vary in both space and time; it only need satisfy:

$$\frac{\partial \tilde{\rho}}{\partial t} = -\tilde{u}_j \frac{\partial \tilde{\rho}}{\partial x_j} \quad (2.29)$$

And, of course, it does this automatically if we insist that $\partial \tilde{u}_k / \partial x_k = 0$. So what's the problem?

The problem is this: What do we choose for the density? Do we really have to know the density at every point in space and time, and use it to solve equation 2.28? This is clearly very complicated, at least for theoretician. And if so, where do we get the variable density from? The answer to the first question is yes, but some simplifications are possible. The answer to the second question takes us directly into either thermodynamics or chemistry, depending on what is causing our density variations. We will consider only the thermodynamic aspects in the next chapter, but for now let's focus on how we can simplify the momentum

equations to make some sense out of them. The first apparently to recognize how to do this was Boussinesq (who is honored with a plaque in the lab where I worked for a while at LML/Lille), hence the name Boussinesq approximation.

Let's imagine the density field to be made up of two parts: one a reference density (which could in fact be the mean density either for the whole field or at a point); and the other part the deviations from the reference value. We can represent this symbolically as:

$$\tilde{\rho} = \rho_o + \Delta\tilde{\rho} \quad (2.30)$$

Note that the reference density does not have to be constant everywhere, but it should be time-independent. The important thing is that it be chosen to minimize the $\Delta\tilde{\rho}$'s.

Let's substitute these into the momentum equation to obtain:

$$(\rho_o + \Delta\tilde{\rho}) \frac{D\tilde{u}_i}{Dt} = -\frac{\partial\tilde{p}}{\partial x_i} + \frac{\partial\tilde{\tau}_{ji}^{(v)}}{\partial x_j} + (\rho_o + \Delta\tilde{\rho})g_i \quad (2.31)$$

Now to see why we go in the direction we will, consider what happens in a fluid at rest with only the reference density. The momentum equation reduces to the hydrostatic equation given by:

$$0 = -\frac{\partial p_o}{\partial x_i} + \rho_o g_i \quad (2.32)$$

where we have used the subscript 'o' to remind us that this is not the real pressure, but only a part of it. As in any hydrostatic problem with a varying density, the changes in the hydrostatic pressure, p_o , reflect the varying weight of the fluid above it. Most importantly, these changes can have nothing to do with the dynamics of the field (since they are static). Therefore we can define a new pressure, say p' which is the difference between the actual pressure and the 'imposed' hydrostatic pressure; i.e.,

$$\frac{\partial p'}{\partial x_i} = \frac{\partial\tilde{p}}{\partial x_i} + \rho_o g_i \quad (2.33)$$

Now we can substitute this into our momentum equation to obtain:

$$(\rho_o + \Delta\tilde{\rho}) \frac{D\tilde{u}_i}{Dt} = -\frac{\partial p'}{\partial x_i} + \frac{\partial\tilde{\tau}_{ji}^{(v)}}{\partial x_j} + g_i \Delta\tilde{\rho} \quad (2.34)$$

It is important to note that we really haven't assumed anything here, nor have we actually changed the equation nor made any approximations – not yet, anyway. In fact the trick of subtracting off the hydrostatic pressure gradient is exactly the same used by all fluid dynamicists to get rid of the gravity term whenever they analyze aerodynamic or hydrodynamic problems. But things are about to change for us.

What Boussinesq realized first was that small density differences made a huge difference in the last term (it goes away entirely if $\Delta\tilde{\rho}$ is zero). But they make very little difference at all to the acceleration term (the left-hand-side) if the differences are small relative to the reference density; i.e., $\Delta\tilde{\rho} \ll \rho_o$. Therefore it is almost as good to work with the following equations as with the original set:

$$\rho_o \frac{D\tilde{u}_i}{Dt} = -\frac{\partial\tilde{p}'}{\partial x_i} + \frac{\partial\tilde{\tau}_{ji}^{(v)}}{\partial x_j} + g_i\Delta\tilde{\rho} \quad (2.35)$$

or equivalently, dividing by the reference density:

$$\frac{D\tilde{u}_i}{Dt} = -\frac{1}{\rho_o} \frac{\partial\tilde{p}'}{\partial x_i} + \frac{\partial\tilde{\tau}_{ji}^{(v)}}{\partial x_j} + g_i \frac{\Delta\tilde{\rho}}{\rho_o} \quad (2.36)$$

Obviously the only difference between these equations and those for constant density flows is the last term. So it is only in the last term that actual density variations must be accounted for. Like many other approximations in fluid mechanics, this can of course be extended and made more exact by using a perturbation analysis using powers of the density ratio.

While this kind of approximation may not make too much sense to the person solving equations exactly in a computer or doing experiments (letting the flow solve the exact equations for you), it has made all the difference in the world to the theoreticians. I think it is safe to say that without the Boussinesq approximation, no theories of natural convection would have been possible, not even the scaling laws that guide so much of engineering analysis. The increase in complication by including the density on the left-hand-side would have been (and still is) overwhelming. And for sure turbulence modelling would be in even more desperate straits. But even for the person who can solve the exact equations in a computer or lab, solutions using the Boussinesq approximation provide a useful tool for simplifying the physics and evaluating the validity of the results. Also, since all theories are based on the Boussinesq approximation, care should be taken when comparing computations and experiments to them, since the underlying assumptions may not be satisfied.

2.8 Some words of caution

A final word of caution is in order, especially for cfd'ers. One of the underlying problems of fluid mechanics is that we really do not know whether the Navier-Stokes equations are well-posed or not. In fact, there is a million dollar mathematical prize for the person who can prove they are. (Some of my friends actually wake up every morning thinking they have proven it, only to end the day discovering they haven't.) The question they ask is whether the equations we use can be shown to have solutions which are singular. Singularities are unphysical. If such singularities can be shown to exist, then we know there are missing terms and missing physics that must resolve them. For example, the equations

of inviscid flow admit to solutions which have infinitesimal sources and sinks and line vortices. Nature does not; it resolves them with viscosity (or in the case of low temperature helium with quantum mechanics). The search for black holes started with the recognition of mathematical singularities. And the excitement about them is because of the new physics that must be necessary to resolve them.

Now almost everyone in fluid mechanics ignores this problem with the full Navier-Stokes equations. And the reason is that we haven't found any singularities, at least when include all the terms we know about. But we do know that finding stable solutions can sometimes be very difficult, and we have invented lots of tricks to stabilize our computations. On the other hand, the very fact that numerical solutions are so difficult could be one of the clues that there is an underlying problem. But this is not the primary point I am trying to make here.

The point I want to emphasize here is that when we start approximating the fundamental equations (as we did above with the Boussinesq approximation and shall do later), we might be messing them up so that they really don't have well-behaved solutions. This is not so much a problem for theoreticians (who are looking for the role of the most important terms) nor for experimentalist (since the flow always knows what it is doing even if we don't). But it could be a the source of major headaches for computationalists. So be on the alert for differences between the solution techniques required for your truncated or approximated equations and the more complete versions. And if they are different, you probably should place more trust the latter.

Chapter 3

Density, Temperature and Energy

The momentum and mass equations of the previous chapter left us with the problem of determining the density. The Boussinesq approximation assumed that small density variations could be neglected as they affected the acceleration terms, but could not be neglected as they affected the gravitational term. Thus even if the background density field can be specified (or guessed), we still have one more unknown (or dependent variable) (five: \tilde{u}_i, \tilde{p} and $\Delta\tilde{\rho}$) than we do equations (four: three momentum plus continuity). Clearly we need to search elsewhere.

Exactly where we should look is almost entirely determined by what the source of our density variations are. There are actually many possibilities: the most common are temperature, salinity, and mixing of different fluids (including air bubbles in liquid for example). All of these have their corresponding conservation equations, some more complicated than others, all have unique problems associated with them. In these notes we shall confine our attention to temperature.

3.1 Why temperature?

In general it is possible to write the equation-of-state for any substance as $\rho = \rho(\theta, p)$ where θ is the absolute temperature. Taking differentials yields:

$$d\rho = \rho\alpha dp - \rho\beta d\theta \quad (3.1)$$

where α is the isothermal compressibility coefficient defined by:

$$\alpha = \left[\frac{1}{\rho} \frac{\partial \rho}{\partial p} \right]_{\theta} \quad (3.2)$$

and β is the isobaric thermal expansivity coefficient defined by:

$$\beta = -\frac{1}{\rho} \left[\frac{\partial \rho}{\partial \theta} \right]_p \quad (3.3)$$

Thus the leading term in a Taylor expansion of the density about a reference state, say (p_o, T_o, ρ_o) is given by:

$$\rho = \rho_o + \rho_o\alpha_o[p - p_o] - \rho_o\beta_o[\theta - \theta_o] + \dots \quad (3.4)$$

where \dots stands for the higher order derivative terms in the Taylor expansion.

In many cases of interest to natural convection, the contribution of the second term on the right-hand-side is orders of magnitude less than the first term. The examples below illustrate this point. Nonetheless, these approximations should be checked in each and every application to see if they apply. Assuming they do, we can approximate the density difference from our reference state by:

$$\Delta\tilde{\rho} \approx -\rho\beta\Delta\tilde{\theta} \quad (3.5)$$

or

$$\frac{\Delta\tilde{\rho}}{\rho_o} \approx -\beta\Delta\tilde{\theta} \quad (3.6)$$

where $\Delta\tilde{\theta}$ is defined as:

$$\Delta\tilde{\theta} = \tilde{\theta} - \theta_o \quad (3.7)$$

3.1.1 Thermal expansion of ideal gases

For an ideal gas, since $p = \rho R\theta$, it is easy to show that:

$$\alpha = \left[\frac{1}{\rho} \frac{\partial \rho}{\partial p} \right]_{\theta} = \frac{1}{p} \quad (3.8)$$

$$\beta = - \left[\frac{1}{\rho} \frac{\partial \rho}{\partial \theta} \right]_p = \frac{1}{\theta} \quad (3.9)$$

Thus from equation 3.4, it follows that for ideal gases:

$$\frac{\Delta\tilde{\rho}}{\rho_o} = \frac{\tilde{p} - p_o}{p_o} - \frac{\Delta\tilde{\theta}}{\theta_o} \quad (3.10)$$

In a typical laboratory experiment (or even in the atmosphere near sea level), the reference pressure is about 1 atm (or $10^5 n/m^2$), the density is about $1.2 kg/m^3$, and the absolute temperature is about $300^\circ K$. The pressure differences arising from the motion are typically of order $\rho_o|u_i|^2$ and velocities above a few meters per second are uncommon. So $\rho_o|u_i|^2$ is of order $10 n/m^2$ which is at least 4 orders of magnitude less than the reference pressure. Thus, as we might have hoped, the second term dominates. But care should be taken when values of $\Delta\tilde{\theta}$ exceed 30° since the second order term in the temperature expansion may be necessary.

Exercise Thermal expansion of liquids

Carry out the same analysis for water at 20 degree C and atmospheric pressure.

3.1.2 A logical inconsistency

In chapter 2 (and in the remainder of the text) we stated that most natural convection flows could be treated as though they were incompressible (i.e., $D\tilde{\rho}/Dt = 0$). But the whole source of thermal buoyancy depends on the fact that the density changes are primarily a result of the temperature changes. Clearly we cannot have $D\tilde{\rho}/Dt = 0$ without also having $D\tilde{\theta}/Dt = 0$. One way to deal with this is to argue that the departures from incompressibility and the isothermal state are small. But this is rather glib, especially since it is the small density (and temperature differences) that are driving our natural convection flow. Fortunately this is more of an intellectual dilemma for laminar flow analysis (and perhaps stability analysis) than it is for turbulence. The reason will be clear later, but it turns out that except for the smallest diffusive scales of the turbulence and very near walls, the condition $D\tilde{\theta}/Dt = 0$ is satisfied.

3.2 The first law of equilibrium thermodynamics

It is the equation of state that leads us to thermodynamics; in particular an equation relating the density to measurable properties, usually temperature and pressure. The pressure is in our list already, but temperature introduces yet another variable. So of course we need yet another equation. And we find it in the first law of thermodynamics.

The whole idea of equilibrium thermodynamics seems at first glance to be completely incompatible with the concept of a fluid in motion. The laws of thermodynamics (as distinguished from the equations derived from them) apply only to fixed quantities of mass of uniform properties throughout and only between equilibrium states. For example if e is the specific energy, we can write:

$$e_2 - e_1 = q_{1 \rightarrow 2} - w_{1 \rightarrow 2} \quad (3.11)$$

where e_1 and e_2 are the energies of equilibrium states 1 and 2 respectively, while $q_{1 \rightarrow 2}$ is the heat added per unit mass and $w_{1 \rightarrow 2}$ is the work done. Even if we split off things like kinetic energy and potential energy and deal only with internal energy we are still left with the problem of how to deal with the non-equilibrium of things we would like to describe. We can of course introduce the concept of quasi-equilibrium by assuming the separation of the equilibrium states to be only infinitesimals and the work and heat infinitesimal as well so we can write:

$$de = \delta q - \delta w, \quad (3.12)$$

where the 'd' emphasizes that e is a state property and path independent while δq and δw are neither. Even so, the idea of things happening infinitely slowly seems like a real stretch when applied to fluid motion.

But to our rescue comes the power of our minds; in particular, calculus and the concept of a fluid particle – the continuum hypothesis if you will. We imagine particles so small that their properties can indeed be imagined to be uniform throughout, and to evolve continuously from one state to another as they move through the fluid. This is really about all we need to be able to apply equilibrium thermodynamics individually to the huge aggregates of fluid particles that comprise a material volume. From there the ideas of fluid mechanics (contact forces, body forces, material derivatives, Reynolds transport theorem) take over and out fall separate equations for the kinetic and internal energy. Only the latter is of interest here. Even an equation for the entropy can be deduced. And amazingly they seem to work well under most circumstances (even through shock waves). (In fact in my experience, only in hypersonics and nanophysics do they break down.)

The result for the internal energy per unit mass, say \tilde{i} , alone is given by:

$$\tilde{\rho} \frac{D\tilde{i}}{Dt} = -\frac{\partial \tilde{q}_n}{\partial x_n} - \tilde{p} \frac{\partial \tilde{u}_m}{\partial x_m} + \tilde{\rho} \tilde{\varepsilon} \quad (3.13)$$

where ε is the viscous dissipation defined by:

$$\tilde{\varepsilon} = 2\mu[\tilde{s}_{ij}\tilde{s}_{ij} - \frac{1}{3}\tilde{s}_{kk}\tilde{s}_{kk}] \quad (3.14)$$

which occurs with opposite sign in the kinetic energy equation. The left-hand side is of course the time rate of change of the internal energy following a fluid particle. The terms on the right-hand-side represent respectively the increase in internal energy due to heat addition (usually down thermal gradients), compression (or decrease due to expansion), and the dissipation of kinetic energy by the distortion of the fluid by the flow.

Alternatively we could choose to work with the corresponding equation for the specific enthalpy, $\tilde{h} = \tilde{i} + \tilde{p}/\tilde{\rho}$ given by:

$$\tilde{\rho} \frac{D\tilde{h}}{Dt} = -\frac{\partial \tilde{q}_n}{\partial x_n} + \frac{D\tilde{p}}{Dt} + \tilde{\rho} \tilde{\varepsilon} \quad (3.15)$$

Since the natural convection flows of interest here will be assumed to be incompressible, the internal energy equation is usually the best choice, since the middle term on the right-hand-side (the compressibility term) vanishes identically.

3.3 Thermal constitutive equations

3.3.1 Fourier fluid

We know that there must be a relation between the instantaneous heat flux vector, \tilde{q}_i , and the instantaneous temperature, $\tilde{\theta}$. Like the relation between viscous stress and strain rate considered earlier, there is no reason *a priori* to consider this relation to be anything particularly simple. The simplest of all possibilities would

be for a linear instantaneous (no history, local in space and time) relationship between heat flux vector and temperature gradient that is independent of direction in the medium (i.e., isotropic medium). It would look like this:

$$\tilde{q}_i = -k \frac{\partial \tilde{\theta}}{\partial x_i} \quad (3.16)$$

Almost unbelievably, almost all fluids of interest behave this way. We call such a fluid a *Fourier* fluid. Or if the fluid is also Newtonian, a *Newtonian-Fourier* fluid. Like the viscosity, k is also a function of the temperature.

It follows immediately that the term in either our internal energy or enthalpy equations involving the divergence of the heat flux can be replaced by:

$$\frac{\partial \tilde{q}_n}{\partial x_n} = -\frac{\partial}{\partial x_n} \left[k \frac{\partial \tilde{\theta}}{\partial x_n} \right] \quad (3.17)$$

Only if k is independent of temperature can we write:

$$\frac{\partial \tilde{q}_n}{\partial x_n} = -k \frac{\partial^2 \tilde{\theta}}{\partial x_n^2} \quad (3.18)$$

Usually it is not, but it is almost always necessary for theorists to assume so anyway. Clearly this is not a requirement for numerical analysis, at least unless the averaged equations of turbulence are used. Nor is it what happens in experiments.

3.3.2 Relating internal energy and enthalpy to temperature

The relations among internal energy, enthalpy and temperature is actually one of the more subtle points of problems involving natural convection, or for that matter all problems involving heat transfer. What we are looking for are the relations involving ‘specific heats’? There are two problems: first there are in principle many different ‘specific heats’; and second even the two we usually work with are in general functions of more than just temperature alone.

Let’s begin by noting that for any simple substance the equation of state relates three independent intensive thermodynamic properties. The two we will be the most interested in are $i = i(\theta, v)$ and $h = h(\theta, p)$ where θ is the absolute temperature and $v = 1/\rho$ is the specific volume. Taking differentials of both yields:

$$di = C_v d\theta + \left[\frac{\partial i}{\partial v} \right]_{\theta} dv \quad (3.19)$$

$$dh = C_p d\theta + \left[\frac{\partial i}{\partial p} \right]_{\theta} dp \quad (3.20)$$

where we have defined:

$$C_v = \left[\frac{\partial i}{\partial \theta} \right]_v \quad (3.21)$$

$$C_p = \left[\frac{\partial h}{\partial \theta} \right]_p \quad (3.22)$$

Note that C_v is usually called the specific heat at constant volume, and is in general a function of both temperature and specific volume (or density). Similarly, C_p is usually called the specific heat at constant pressure, and is in general a function of both temperature and pressure. Interestingly (and quite usefully) it can be shown from the wonders of calculus that C_v and C_p are directly related to the quantities we defined earlier, α and β through the so-called Mayer relation given by:

$$C_p - C_v = \frac{\theta \beta^2}{\rho \alpha} \quad (3.23)$$

For an ideal gas, for example, it is quite easy to show from this equation that $C_p - C_v = R$, the gas constant. It is also possible to show that for a liquid that $C_p = C_v = C$, if the liquid is assumed to be incompressible. We shall use these relations below.

It is easy to see that if a process is a constant volume process, then $di = C_v(\theta, v)d\theta$ and if it is a constant pressure process, then $dh = C_p(\theta, p)d\theta$. BUT these are very special processes, and not true in general. Therefore in order to use these relations in equations 3.13 and 3.29, we must use the general forms of equations 3.19 and 3.20 respectively. By the same arguments we presented above for applying equilibrium thermodynamics to a moving fluid, it follows that:

$$\frac{D\tilde{i}}{Dt} = C_v \frac{D\tilde{\theta}}{Dt} + \left[\frac{\partial i}{\partial v} \right]_{\theta} \frac{D}{Dt} \left(\frac{1}{\tilde{\rho}} \right) \quad (3.24)$$

$$\frac{D\tilde{h}}{Dt} = C_p \frac{D\tilde{\theta}}{Dt} + \left[\frac{\partial h}{\partial p} \right]_{\theta} \frac{D\tilde{p}}{Dt} \quad (3.25)$$

Also, although we have not given them names, the partial derivatives in square bracket terms of equations 3.19 and 3.20 (the second term on the right-hand-side of each) are also thermodynamic properties and functions of the their respective variables. As will be seen below, they turn out to be quite important in deriving the various forms of the internal energy transport equation.

3.3.3 Ideal gases

In general the internal energy and enthalpy are state properties themselves, so in a simple fluid (single chemical composition) they must be determined from two other state properties. Happily most of the flows we encounter exist at temperatures well above the critical point and at pressures which are much lower, thus the gases

we encounter (usually air) behave as ideal gases. The whole idea of an ideal gas is that its internal energy is a function of temperature, say θ , only; i.e., $di = C_v(\theta)d\theta$ where C_v is the specific heat at constant volume which is defined by:

$$C_v(\theta) \equiv \left. \frac{\partial i}{\partial \theta} \right|_v = \frac{di}{d\theta} \quad (3.26)$$

Even more fortunate for us, the range of temperatures involved (at least if there is no combustion) is small enough that we can often pretend that the gas is calorically perfect; i.e., $C_v = C_{vo} = \text{constant}$.

Whether calorically perfect or only simply perfect, equation 3.26 can be immediately substituted into the internal energy equation to obtain the familiar temperature equation of most heat transfer courses as:

$$\tilde{\rho}C_v \frac{D\tilde{\theta}}{Dt} = -\frac{\partial \tilde{q}_k}{\partial x_k} + \tilde{p} \frac{\partial \tilde{u}_m}{\partial x_m} + \tilde{\rho}\tilde{\varepsilon} \quad (3.27)$$

We can derive a similar form using the enthalpy. For an ideal gas, $p = \rho R\theta$ where R is the gas constant. The specific heat at constant pressure is defined by convention as:

$$C_p \equiv \left. \frac{\partial h}{\partial \theta} \right|_\theta \quad (3.28)$$

But clearly if the gas is ideal then C_p is itself a function of temperature only and in fact, $dh = C_p d\theta$, Also $R = C_p - C_v$. And if the gas may be assumed to be calorically perfect, then $C_p = C_{po}$ is a constant. Substitution of equation 3.28 into equation 3.29 yields directly:

$$\tilde{\rho}C_p \frac{D\tilde{\theta}}{Dt} = -\frac{\partial \tilde{q}_k}{\partial x_k} + \frac{D\tilde{p}}{Dt} + \tilde{\rho}\tilde{\varepsilon} \quad (3.29)$$

3.3.4 Liquids

The specific heats for liquids are somewhat simpler since they are generally considered to be incompressible. If they can be further assumed to be of constant density, then by application of Maxwell's relations it is possible to derive Mayer's relation from which it follows immediately that for an incompressible fluid $C_p = C_v = C(T)$ only. In the internal energy equation the divergence term on the right-hand-side vanishes due to the assumed incompressibility so the result is:

$$\tilde{\rho}C \frac{D\tilde{\theta}}{Dt} = -\frac{\partial \tilde{q}_k}{\partial x_k} + \tilde{\rho}\tilde{\varepsilon} \quad (3.30)$$

BUT the enthalpy in a liquid is *not* a function of temperature only. Therefore for the enthalpy the more general form of equation 3.25 must be used. Substitution into the enthalpy equation requires a bit more manipulation using the general thermodynamic identity:

$$\left[\frac{\partial h}{\partial p} \right]_{\theta} = \left\{ -\theta \left[\frac{\partial v}{\partial \theta} \right]_p + v \right\} \quad (3.31)$$

If the liquid is assumed to be of constant density, then $v = 1/\rho = \text{constant}$ so the first term in equation is identically zero and only the last survives. This when multiplied by the pressure cancels the Dp/Dt term to yield the same result as equation 3.30.

But it is easy to see that the logic of the preceding paragraph appears to break down if it is the change in density of a liquid with temperature which is driving the flow, since then the first term of equation 3.31 can not be assumed to be zero. So the question must be raised: ‘What does it mean in equation 3.31 that $v = \text{constant}$. This is where the arguments of the third paragraph of section 3.2 come into play. The only specific volume that must be constant is that of the particular thermodynamic system to which we are applying the laws of thermodynamics, which in our case is the fluid particle. This is of course satisfied if the time derivative following the fluid motion of the specific volume is zero; i.e., $Dv/Dt = 0$. This is of course is satisfied if $D\tilde{\rho}/Dt = 0$. In other words: the real condition for the validity of our thermodynamic hypotheses above is incompressibility.

For natural convection in liquids it is often the temperature which is responsible for the density variations. In other words, the change in specific volume or density with temperature. But this is precisely the term we have just assumed to be zero. Both conditions can be satisfied (I think) only if $D\theta/Dt = 0$, but this doesn’t really seem to make sense either — unless we further assume there is no conduction and the particles move adiabatically. In fact one can make just such an argument for high Reynolds number turbulence, where the viscous and diffusion effects are confined to the very smallest scales of the turbulence, leaving most the turbulence scales effectively inviscid (and non-diffusive).

3.4 The temperature equation

We now have all the pieces we need to write down an equation governing the temperature field for natural convection flows (and many other flows as well). As pointed out earlier, the analysis of most natural convection flows begins with the assumption of the incompressible motion. Also, most of our fluids behave as a Newtonian/Fourier fluid. Also the dissipation of kinetic energy is almost always negligible relative to the other terms in the internal energy and enthalpy equations.

3.4.1 Ideal gases

Thus for ideal gases, either of the following forms of the equations can be used:

$$\tilde{\rho}\tilde{C}_v \frac{D\tilde{\theta}}{Dt} = \frac{\partial}{\partial x_n} \left[k \frac{\partial \tilde{\theta}}{\partial x_n} \right] \quad (3.32)$$

or

$$\tilde{\rho}\tilde{C}_p \frac{D\tilde{\theta}}{Dt} = \frac{\partial}{\partial x_n} \left[k \frac{\partial \tilde{\theta}}{\partial x_n} \right] + \frac{Dp}{Dt} \quad (3.33)$$

It is easy to see that these are the same for an incompressible flow by using $p = \rho R\theta$ and $R = C_p - C_v$.

Often it still be necessary (at least for theoreticians) to assume the thermal conductivity and density to be constant as well, say k_o and ρ_o respectively. This is never really true, but the assumptions make solutions possible. It is easy to spot when these assumptions have been made since the equations are usually written in the following forms:

$$\frac{D\tilde{\theta}}{Dt} = \kappa_v \frac{\partial^2 \tilde{\theta}}{\partial x_n^2} \quad (3.34)$$

where κ_v is defined by:

$$\kappa_v = \frac{k}{\rho_o C_v} \quad (3.35)$$

or

$$\frac{D\tilde{\theta}}{Dt} = \kappa_p \frac{\partial^2 \tilde{\theta}}{\partial x_n^2} + \left[\frac{1}{\rho C_p} \right] \frac{Dp}{Dt} \quad (3.36)$$

where κ_p is defined by:

$$\kappa_p = \frac{k}{\rho_o C_p} \quad (3.37)$$

3.4.2 Liquids

For liquids, $C_p = C_v = C$, which together with other thermodynamic relations imply that these simplify to:

$$\tilde{\rho}C \frac{D\tilde{\theta}}{Dt} = \frac{\partial}{\partial x_k} \left[k \frac{\partial \tilde{\theta}}{\partial x_n} \right] \quad (3.38)$$

Again it is customary (at least theoretically) to assume the thermal conductivity is a constant so we can write:

$$\frac{D\tilde{\theta}}{Dt} = \kappa \frac{\partial^2 \tilde{\theta}}{\partial x_n^2} \quad (3.39)$$

where

$$\kappa = \frac{k}{\rho C} \quad (3.40)$$

3.5 Boussinesq-‘friendly’ forms of the equations

We can use equation 3.6 to rewrite the Boussinesq form of our momentum equation given by equation 2.36 as:

$$\frac{D\tilde{u}_i}{Dt} = -\frac{1}{\rho_o} \frac{\partial \tilde{p}'}{\partial x_i} + \frac{\partial \tilde{\tau}_{ji}^{(v)}}{\partial x_j} - g_i \beta \Delta \tilde{\theta} \quad (3.41)$$

where $\Delta \tilde{\theta} = \tilde{\theta} - \theta_o$ and β is usually evaluated at the reference state.

As is clear from the equation above, in natural convection problems it is seldom the temperature field itself which is of interest, but rather $\Delta \tilde{\theta}$, the temperature differences from a steady reference state. We can recast the energy equations above in the more convenient form given by:

$$\frac{D}{Dt} \Delta \tilde{\theta} = \kappa \frac{\partial^2}{\partial x_n^2} \Delta \tilde{\theta} + \left\{ \kappa \frac{\partial^2 \theta_o}{\partial x_n^2} - u_j \frac{\partial \theta_o}{\partial x_j} \right\} \quad (3.42)$$

If the reference state can be chosen to be uniform (i.e., \vec{x} independent), then the terms in brackets on the right-hand-side are identically zero. But sometimes, especially in experiments, the reference state is not uniform due to an overall stratification in the facility. In such cases, the conduction term is still usually negligible. But it is very important to include the last term since even small gradients in θ_o can substantially affect the flow by adding or stealing buoyancy from it.

Alternatively, if we agree that β is to be a constant, we can multiply equation 3.43 by $-g_i \beta$ to produce a transport equation for the buoyancy directly as:

$$\frac{D}{Dt} [g_i \beta \Delta \tilde{\theta}] = \kappa \frac{\partial^2}{\partial x_n^2} [g_i \beta \Delta \tilde{\theta}] + \left\{ \left[\kappa \frac{\partial^2}{\partial x_n^2} - u_j \frac{\partial}{\partial x_j} \right] g_i \beta \theta_o \right\} \quad (3.43)$$

This form will be seen to be particularly useful for evaluating integral constraints for conservation of overall momentum and buoyancy.

3.6 Thermal boundary conditions

Thermal boundary conditions fall into two groups: one where the wall temperature is specified, the other where the wall heat flux is specified. For a Fourier fluid the latter establishes the relation between the heat flux and the temperature gradient at the wall. In n_i is the unit outward normal to the fluid at the wall, the rate at which is heat is *added* to the flow is given by:

$$\tilde{q}_i n_i = -k \frac{\partial \tilde{\theta}}{\partial x_i} \Big|_w n_i \quad (3.44)$$

In most problems (aside from idealized ones), neither the wall temperature nor the wall heat flux are constant, and both vary along the surface. This is a

much more difficult problem, and one often makes a local approximation that one or the other is constant. The alternative is to solve the internal wall conduction problem in parallel, usually numerically.

Chapter 4

The Reynolds Averaged Equations and the Turbulence Closure Problem

This chapter has been adapted from the author's lecture notes entitled "Lectures on Turbulence for the 21st Century". They are available on the website

www.turbulence-online.com

and are updated occasionally. This chapter is incomplete, missing the equations for the turbulent heat fluxes.

4.1 Equations for the Average Velocity

Turbulence is that chaotic state of motion characteristic of solutions to the equations of motion at high Reynolds number. Although laminar solutions to the equations often exist that are consistent with the boundary conditions, perturbations to these solutions (sometimes even infinitesimal) can cause them to become turbulent. To see how this can happen, it is convenient to analyze the flow in two parts, a mean (or average) component and a fluctuating component. Thus the instantaneous velocity and stresses can be written as:

$$\begin{aligned}\tilde{u}_i &= U_i + u_i \\ \tilde{p} &= P + p \\ \tilde{\tau}_{ji}^{(v)} &= T_{ij}^{(v)} + \tau_{ji}^{(v)} \\ \Delta\tilde{\rho} &= \langle \Delta\tilde{\rho} \rangle + \Delta\rho\end{aligned}\tag{4.1}$$

where U_i , p , and $T_{ji}^{(v)}$ represent the mean motion, and u_i , p , and τ_{ji} the fluctuating motions. $\langle \rangle$ represents the ensemble average, so that $U_i = \langle \tilde{u}_i \rangle$, etc. This

technique for decomposing the instantaneous motion is referred to as the *Reynolds decomposition*. Note that if the averages are defined as ensemble means, they are, in general, *time-dependent*. Mean and fluctuating quantities for the temperature and heat flux can be similarly defined: i.e.

$$\tilde{q}_i = Q_i + q_i \quad (4.2)$$

$$\tilde{\theta} = \Theta + \theta$$

$$\Delta\tilde{\theta} = \langle\Delta\tilde{\theta}\rangle + \Delta\theta \quad (4.3)$$

In this and succeeding chapters we shall work only with equations for which the Boussinesq approximation can be made. Thus the density will be assumed constant in every term *except for when it appears as a buoyancy term*. These will be easy to spot since they will involve the gravitation parameter, g , explicitly.

Substitution of equations 4.1 into equations 2.36 yields

$$\frac{\partial(U_i + u_i)}{\partial t} + (U_j + u_j) \frac{\partial(U_i + u_i)}{\partial x_j} = -\frac{1}{\rho_o} \frac{\partial(P + p)}{\partial x_i} + \frac{1}{\rho_o} \frac{\partial(T_{ji}^{(v)} + \tau_{ji}^{(v)})}{\partial x_j} + \frac{g_i}{\rho_o} [\langle\Delta\rho\rangle + \Delta\rho] \quad (4.4)$$

This equation can now be averaged to yield an equation expressing momentum conservation for the averaged motion. Note that the operations of averaging and differentiation commute; i.e., the average of a derivative is the same as the derivative of the average. Also, the average of a fluctuating quantity is zero.¹ Thus the equation for the averaged motion reduces to:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{1}{\rho} \frac{\partial T_{ji}^{(v)}}{\partial x_j} - \langle u_j \frac{\partial u_i}{\partial x_j} \rangle + \frac{g_i}{\rho_o} \langle \Delta\rho \rangle \quad (4.5)$$

where the remaining fluctuating product term has been moved to the right-hand side of the equation. Whether or not this last term is zero like the other fluctuating terms depends on the correlation of terms in the product. In general, these correlations are *not* zero.

The mass conservation equation can be similarly decomposed. In incompressible form, substitution of equations 4.1 into equation 2.16 yields:

$$\frac{\partial(U_j + u_j)}{\partial x_j} = 0 \quad (4.6)$$

of which the average is:

$$\frac{\partial U_j}{\partial x_j} = 0 \quad (4.7)$$

It is clear from equation 4.7 that the averaged motion satisfies the same form of the mass conservation equation as does the instantaneous motion, at least

¹These are easily proven from the definitions of both.

for incompressible flows. How much simpler the turbulence problem would be if the same were true for the momentum! Unfortunately, as is easily seen from equation 4.5, such is not the case.

Equation 4.7 can be subtracted from equation 4.6 to yield an equation for the instantaneous motion alone; i.e.,

$$\frac{\partial u_j}{\partial x_j} = 0 \quad (4.8)$$

Again, like the mean, the form of the original instantaneous equation is seen to be preserved. The reason, of course, is obvious: the continuity equation is linear. The momentum equation, on the other hand, is not; hence the difference.

Equation 4.8 can be used to rewrite the last term in equation 4.5 for the mean momentum. Multiplying equation 4.8 by u_i and averaging yields:

$$\langle u_i \frac{\partial u_j}{\partial x_j} \rangle = 0 \quad (4.9)$$

This can be added to $\langle u_j \partial u_i / \partial x_j \rangle$ to obtain:

$$\langle u_j \frac{\partial u_i}{\partial x_j} \rangle + 0 = \langle u_j \frac{\partial u_i}{\partial x_j} \rangle + \langle u_i \frac{\partial u_j}{\partial x_j} \rangle = \frac{\partial}{\partial x_j} \langle u_i u_j \rangle \quad (4.10)$$

where again the fact that arithmetic and averaging operations commute has been used.

The equation for the averaged momentum, equation 4.5 can now be rewritten as:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho_o} \frac{\partial P}{\partial x_i} + \frac{1}{\rho_o} \frac{\partial T_{ji}^{(v)}}{\partial x_j} - \frac{\partial}{\partial x_j} \langle u_i u_j \rangle + \frac{g_i}{\rho_o} \langle \Delta \rho \rangle \quad (4.11)$$

The last two terms on the right-hand side are both divergence terms and can be combined; the result is:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho_o} \frac{\partial P}{\partial x_i} + \frac{1}{\rho_o} \frac{\partial}{\partial x_j} \left[T_{ji}^{(v)} - \rho_o \langle u_i u_j \rangle \right] + \frac{g_i}{\rho_o} \langle \Delta \rho \rangle \quad (4.12)$$

Now the terms in square brackets on the right have the dimensions of stress. The first term is, in fact, the *viscous stress*. The second term, on the other hand, is not a stress at all, but simply a re-worked version of the fluctuating contribution to the non-linear acceleration terms. The fact that it can be written this way, however, indicates that *at least as far as the mean motion is concerned*, it *acts* as though it were a *stress* — hence its name, the **Reynolds stress**. In the succeeding sections the consequences of this difference will be examined.

4.2 The ‘Temperature’ Equation

In section 3.1 We clearly need to have a means to independently produce the mean density differences, $\langle \Delta \rho \rangle$, from our assumed reference density field, ρ_o . In general,

ρ_o itself could vary in space, but it will considerably complicate our selection of the values of thermal diffusivity and kinematic viscosity if we do. Since our goal in this section is not to set up equations for the numerical analysis of laminar flows, we will just assume at the outset the ρ_o is the same everywhere; i.e., space-independent. Thus it is entirely likely that the equations we derive might not be the best choice for any particular problem.

$$\frac{\partial \Theta}{\partial t} + U_j \frac{\partial \Theta}{\partial x_j} = -\langle u_j \frac{\partial \theta}{\partial x_j} \rangle + \frac{1}{\rho_o C_v} \frac{\partial}{\partial x_j} \left[k \frac{\partial \Theta}{\partial x_j} \right] \quad (4.13)$$

where we have neglected the contribution from the mechanical dissipation of energy. The last first two terms on the right-hand side are both divergence terms and can be combined. Using the fluctuating continuity equation as above this can be written as:

$$\frac{\partial \Theta}{\partial t} + U_j \frac{\partial \Theta}{\partial x_j} = -\frac{\partial}{\partial x_j} \langle \theta u_j \rangle + \frac{1}{\rho_o C_v} \frac{\partial}{\partial x_j} \left[k \frac{\partial \Theta}{\partial x_j} \right] \quad (4.14)$$

If we can assume ρ_o and C_v to be constant, we can combine the last two terms to obtain:

$$\frac{\partial \Theta}{\partial t} + U_j \frac{\partial \Theta}{\partial x_j} = \frac{\partial}{\partial x_j} \left[-\langle \theta u_j \rangle + \frac{k}{\rho_o C} \frac{\partial \Theta}{\partial x_j} \right] \quad (4.15)$$

Now the role of the term $-\langle \theta u_j \rangle$ as representing the turbulent heat fluxes is quite obvious. As will be seen below, like the Reynolds stress, the presence of this term considerably complicates our problem.

4.3 The Turbulence Problem

It is the appearance of the Reynolds stress which makes the turbulence problem so difficult — at least from the engineers perspective. Even though we can pretend it is a stress, the physics which give rise to it are very different from the viscous stress. The viscous stress can be related directly to the other flow properties by constitutive equations, which in turn depend only on the properties of the *fluid* (as in equation ?? for a Newtonian fluid). The reason this works is that when we make such closure approximations for a fluid, we are averaging over characteristic length and time scales much smaller than those of the *flows* we are interested in. Yet at the same time, these scales are much larger than the *molecular* length and time scales which characterize the molecular interactions that are actually causing the momentum transfer. (This is what the continuum approximation is all about.)

The *Reynolds stresses*, on the other hand, arise directly from the *flow* itself! Worse, the scales of the fluctuating motion which give rise to it **are** the scales we are interested in. This means that the closure ideas which worked so well for the

viscous stress, should not be expected to work too well for the Reynolds stress. And as we shall see, they do not.

This leaves us in a terrible position. Physics and engineering are all about writing equations (and boundary conditions) so we can solve them to make predictions. We don't want to have to build prototype airplanes first to see if they will fall out of the sky. Instead we want to be able to analyze our designs *before* building, to save the cost in money and lives if our ideas are wrong. The same is true for dams and bridges and tunnels and automobiles. If we had confidence in our turbulence models, we could even build huge one-offs and expect them to work the first time. Unfortunately, even though turbulence models have improved to the point where we can use them in design, we still cannot trust them enough to eliminate expensive wind tunnel and model studies. And recent history is full of examples to prove this.

The turbulence problem (from the engineers perspective) is then three-fold:

- **The averaged equations are not closed.** Count the unknowns in equation 4.12 above. Then count the number of equations. Even with the continuity equation we have at least six equations too few.
- **The simple ideas to provide the extra equations usually do not work.** And even when we can fix them up for a particular class of flows (like the flow in a pipe, for example), they will most likely not be able to predict what happens in even a slightly different environment (like a bend).
- **Even the last resort of compiling engineering tables for design handbooks carries substantial risk.** This is the last resort for the engineer who lacks equations or cannot trust them. Even when based on a wealth of experience, they require expensive model testing to see if they can be extrapolated to a particular situation. Often they cannot, so infinitely clever is Mother Nature in creating turbulence that is unique to a particular set of boundary conditions.

Turbulent flows are indeed flows! And that is the problem.

4.4 The Origins of Turbulence

Turbulent flows can often be observed to arise from laminar flows as the Reynolds number, (or some other relevant parameter) is increased. This happens because small disturbances to the flow are no longer damped by the flow, but begin to grow by taking energy from the original laminar flow. This natural process is easily visualized by watching the simple stream of water from a faucet (or even a pitcher). Turn the flow on very slowly (or pour) so the stream is very smooth initially, at least near the outlet. Now slowly open the faucet (or pour faster) and observe what happens, first far away, then closer to the spout. The surface

begins to exhibit waves or ripples which appear to grow downstream. In fact, they are growing by extracting energy from the primary flow. Eventually they grow enough that the flow breaks into drops. These are capillary instabilities arising from surface tension, but regardless of the type of instability, the idea is the same — small (or even infinitesimal) disturbances have grown to disrupt the serenity (and simplicity) of laminar flow.

The manner in which instabilities grow naturally in a flow can be examined using the equations we have already developed above. We derived them by decomposing the motion into a mean and a fluctuating part. But suppose instead we had decomposed the motion into a *base* flow part (the initially laminar part) and into a *disturbance* which represents a fluctuating part superimposed on the base flow. The result of substituting such a decomposition into the full Navier-Stokes equations and averaging is precisely that given by equations 4.5 and 4.7. But the very important difference is the additional restriction that what was previously identified as *the mean (or averaged) motion is now also the base or laminar flow*.

Now if the base flow is really a laminar flow (which it must be by our original hypothesis), then our averaged equations governing the base flow must yield the same mean flow solution as the original laminar flow on which the disturbance was superimposed. But this can happen only if these new averaged equations reduce to **exactly** the same laminar flow equations without any evidence of a disturbance. Clearly from equations 4.5 and 4.7, this can happen *only if all the Reynolds stress terms vanish identically!* Obviously this requires that the disturbances be infinitesimal so the extra terms can be neglected — hence our interest in infinitesimal disturbances.

So we hypothesized a base flow which was laminar and showed that it is unchanged even with the imposition of infinitesimal disturbances on it — *but only as long as the disturbances remain infinitesimal!* What happens if the disturbance starts to grow? Obviously before we conclude that all laminar flows are laminar forever we better investigate whether or not these infinitesimal disturbances can grow to *finite* size. To do this we need an equation for the fluctuation itself.

An equation for the fluctuation (which might be an imposed disturbance) can be obtained by subtracting the equation for the mean (or base) flow from that for the instantaneous motion. We already did this for the continuity equation. Now we will do it for the momentum equation. Subtracting equation 4.5 from equation 4.1 yields an equation for the fluctuation as:

$$\rho \left[\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} \right] = - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ji}^{(v)}}{\partial x_j} - \rho \left[u_j \frac{\partial U_i}{\partial x_j} \right] - \rho \left\{ u_j \frac{\partial u_i}{\partial x_j} - \langle u_j \frac{\partial u_i}{\partial x_j} \rangle \right\} \quad (4.16)$$

It is very important to note the type and character of the terms in this equation. First note that the left-hand side is the derivative of the *fluctuating* velocity following the *mean* motion. This is exactly like the term which appears on the left-hand side of the equation for the mean velocity, equation 4.5. The first two terms on the right-hand side are also like those in the mean motion, and represent

the fluctuating pressure gradient and the fluctuating viscous stresses. The third term on the right-hand side is new, and will be seen later to represent the primary means by which fluctuations (and turbulence as well!) extract energy from the mean flow, the so-called *production terms*. The last term is quadratic in the fluctuating velocity, unlike all the others which are linear. Note that all of the terms vanish identically if the equation is averaged, the last because its mean is subtracted from it.

Now we want to examine what happens if the disturbance is small. In the limit as the amplitude of the disturbance (or fluctuation) is *infinitesimal*, the bracketed term in the equation for the fluctuation vanishes (since it involves products of infinitesimals), and the remaining equation is *linear in the disturbance*. The study of whether or not such infinitesimal disturbances can grow is called **Linear Fluid Dynamic Stability Theory**. These linearized equations are very different from those governing turbulence. Unlike the equations for disturbances of *finite* amplitude, the linearized equations are well-posed (or closed) since the Reynolds stress terms are gone.

The absence of the non-linear terms, however, constrains the validity of the linear analysis to only the initial stage of disturbance growth. This is because as soon as the fluctuations begin to grow, their amplitudes can no longer be assumed infinitesimal and the Reynolds stress (or more properly, the non-linear fluctuating terms), become important. As a result the base flow equations begin to be modified so that the solution to them can no longer be identical to the laminar flow (or base flow) from which it arose. Thus while linear stability theory can predict *when* many flows become *unstable*, it can say very little about *transition to turbulence* since this process is highly non-linear.

It is also clear from the above why the process of transition to turbulence is so dependent on the state of the background flow. If the disturbances present in the base flow are small enough, then Linear Stability Theory will govern their evolution. On the other hand if the disturbances to the base flow are not small enough, Linear Stability Theory can never apply since the non-linear terms will never be negligible. This is so-called *by-pass transition*. It is not uncommon to encounter situations like this in engineering environments where the incoming flow has a modest turbulence level super-imposed upon it. In such cases, the nature of the disturbances present is as important as their intensities, with the consequence that a general transition criterion may not exist, and perhaps should not even be expected.

4.5 The importance of non-linearity

We saw in the preceding section that non-linearity was one of the essential features of turbulence. When small disturbances grow large enough to interact *with each other*, we enter a whole new world of complex behavior. Most of the rules we learned for linear systems do not apply. Since most of your mathematical training

has been for linear equations, most of your mathematical intuition therefore will not apply either. On the other hand, you may surprise yourself by discovering how much your *non-mathematical* intuition already recognizes non-linear behavior and accounts for it.

Consider the following simple example. Take a long stick with one person holding each end and stand at the corner of a building. Now place the middle of the stick against the building and let each person apply pressure in the same direction so as to bend the stick. If the applied force is small, the stick deflects (or bends) a small amount. Double the force, and the deflection is approximately doubled. Quadruple the force and the deflection is quadrupled. Now you don't need a Ph.D. in Engineering to know what is going to happen if you continue this process. **The stick is going to break!**

But where in the equations for the deflection of the stick is there anything that predicts this can happen? Now if you are thinking only like an engineer, you are probably thinking: he's asking a stupid question. Of course you can't continue to increase the force because you will exceed first the yield stress, then the breaking limit, and of course the stick will break.

But pretend I am the company president with nothing more than an MBA.² I don't know much about these things, but you have told me in the past that your computers have equations to predict everything. So I repeat: Where in the equations for the deflection of this stick does it tell me this is going to happen?

The answer is very simple: **There is *nothing* in the equations that will predict this.** And the reason is also quite simple: You lost the ability to predict catastrophes like breaking when you linearized the fundamental equations — which started out as Newton's Law too. In fact, before linearization, they were exactly the same as those for a fluid, only the constitutive equation was different.

If we had NOT linearized these equations and had constitutive equations that were more general, then we possibly could apply these equation right to and past the limit. The point of fracture would be a bifurcation point for the solution.

Now the good news is that for things like reasonable deflections of beams, linearization works wonderfully since we hope most things we build don't deflect too much — especially if you are sitting on a fault as I am at this moment.³ Unfortunately, as we noted above, for fluids the disturbances tend to quickly become dominated by the non-linear terms. This, of course, means our linear analytical techniques are pretty useless for fluid mechanics, and especially turbulence.

But all is not lost. Just as we have learned to train ourselves to anticipate when sticks break, we have to train ourselves to anticipate how non-linear fluid phenomena behave. Toward that end we will consider two simple examples: one from algebra — the logistic map, and one from fluid mechanics — simple vortex stretching.

²For some reason the famous o-ring disaster of the the Challenger space shuttle comes to mind here.

³I am sitting at the moment of this writing at the Institute for Theoretical Physics at the University of California/Santa Barbara.

Example 1: An experiment with the logistic map.

Consider the behavior of the simple equation:

$$y_{n+1} = ry_n(1 - y_n) \quad (4.17)$$

where $n = 1, 2, \dots$, $0 < y < 1$ and $r > 0$. The idea is that you pick any value for y_1 , use the equation to find y_2 , then insert that value on the right-hand side to find y_3 , and just continue the process as long as you like. Make sure you note any dependence of the final result on the initial value for y .

- First notice what happens if you linearize this equation by disregarding the term in parentheses; i.e., consider the simpler equation $y_{n+1} = ry_n$. My guess is that you won't find this too exciting — unless, of course, you are one of those rare individuals who likes watching grass grow.
- Now consider the full equation and note what happens for $r < 3$, and especially what happens for very small values of r . Run as many iterations as necessary to make sure your answer has converged. Do NOT try to take short-cuts by programming all the steps at once. Do them one at a time so you can see what is happening. Believe me, it will be much easier this way in the long run.
- Now research carefully what happens when $r = 3.1, 3.5$, and 3.8 . Can you recognize any patterns.
- Vary r between 3 and 4 to see if you can find the boundaries for what you are observing.
- Now try values of $r > 4$. How do you explain this?

Example 2: Stretching of a simple vortex.

Imagine a simple vortex filament that looks about like a strand of spaghetti. Now suppose it is in an otherwise steady inviscid incompressible flow. Use the vorticity equation to examine the following:

- Examine first what happens to it in two-dimensional velocity field. Note particularly whether any new vorticity can be produced; i.e., can the material derivative of the vorticity ever be greater than zero? (Hint: look at the $\omega_j \partial u_i / \partial x_j$ -term.)
- Now consider the same vortex filament in a three-dimensional flow. Note particularly the various ways new vorticity can be produced — if you have some to start with! Does all this have anything to do with non-linearities?

Now you are ready for a real flow.

A Simple Experiment: The Starbucks⁴ problem

Go to the nearest coffee pot (or your favorite coffee shop) and get a cup of coffee. (Note that you are not required to drink it, just play with it.) Then slowly and carefully pour a little cream (or half and half, skim milk probably won't work) into it. Now ever so gently, give it a simple single stir with a stick or a spoon and observe the complex display that you see. Assuming that the cream and coffee move together, and that the vorticity (at least for a while) moves like fluid material, explain what you see in the light of Example 2 above.

4.6 The Turbulence Closure Problem and the Eddy Viscosity

From the point of view of the averaged motion, at least, the problem with the non-linearity of the instantaneous equations is that they introduce new unknowns, the Reynolds stress into the averaged equations. There are six individual stress components we must deal with to be exact: $\langle u_1^2 \rangle$, $\langle u_2^2 \rangle$, $\langle u_3^2 \rangle$, $\langle u_1 u_2 \rangle$, $\langle u_1 u_3 \rangle$, and $\langle u_2 u_3 \rangle$. These have to be related to the mean motion itself before the equations can be solved, since the number of unknowns and number of equations must be equal. The absence of these additional equations is often referred to as **the Turbulence Closure Problem**.

A similar problem arose when the instantaneous equations were written, since relations had to be introduced to relate the stresses (in particular, the viscous stresses) to the motion itself. These relations (or constitutive equations) *depended only on the properties of the fluid material, and not on the flow itself*. Because of this fact, it is possible to carry out independent experiments, called viscometric experiments, in which these fluid properties can be determined once and for all. Equation ?? provides an example of just such a constitutive relation, the viscosity, μ , depending only in the choice of the material. For example, once the viscosity of water at given temperature is determined, this value can be used in all flows at that temperature, not just the one in which the evaluation was made.

It is tempting to try such an approach for the turbulence Reynolds stresses (even though we know the underlying requirements of scale separation are not satisfied). For example, a Newtonian type closure for the Reynolds stresses, often referred to as an “eddy” or “turbulent” viscosity model, looks like:

$$-\rho \langle u_i u_j \rangle + \frac{1}{3} \langle u_i u_i \rangle = \mu_t \left[S_{ij} - \frac{1}{3} S_{kk} \delta_{ij} \right] \quad (4.18)$$

⁴Starbucks is a very popular chain of coffee shops in the USA and many other countries who have only recently discovered what good coffee tastes like.

where μ_t is the turbulence “viscosity” (also called the eddy viscosity), and S_{ij} is the *mean* strain rate defined by:

$$S_{ij} = \frac{1}{2} \left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right] \quad (4.19)$$

The second term vanishes identically for incompressible flow. For the simple case of a two-dimensional shear flow, equation 4.18 for the Reynolds shear stress reduces to

$$-\rho \langle u_1 u_2 \rangle = \mu_t \frac{\partial U_1}{\partial x_2} \quad (4.20)$$

Note this “model” is the direct analogy to the Newtonian model for viscous stress in a fluid. The Reynolds stresses, $\langle -u_i u_j \rangle$ replaces the viscous stress, $\tau_{ij}^{(v)}$. The counterpart to the mechanical pressure is the mean normal Reynolds stress, $\langle u_i u_i \rangle / 3$. And like it’s fluid counterpart it, the Reynolds stress can depend only on the mean strain rate at a single instant and single location in the flow, so has no history or non-local dependence. This absence will turn out to be fatal in most applications. Moreover, unlike like the viscosity, μ , which depends only on the fluid and not the motion itself, the “turbulence viscosity”, μ_t , depends entirely on the motion.

That such a simple model can adequately describe the mean motion in at least one flow is illustrated by the axisymmetric buoyant plume sketched in Figure 4.1. Figures 4.2 and 4.3 show the calculation of the mean velocity and temperature profiles respectively. Obviously the mean velocity and temperature profiles are reasonably accurately computed, as are the Reynolds shear stress and lateral turbulent heat flux shown in Figures 4.4 and 4.5.

The success of the eddy viscosity in the preceding example is more apparent than real, however, since the value of the eddy viscosity and eddy diffusivity (for the turbulent heat flux) have been chosen to give the best possible agreement with the data. This, in itself, would not be a problem if that chosen values could have been obtained in advance of the computation, or even if they could be used to successfully predict other flows. In fact, the values used work only for this flow, thus the computation is *not a prediction at all, but a **postdiction** or **hindcast*** from which no extrapolation to the future can be made. In other words, our turbulence “model” is about as useful as having a program to predict yesterday’s weather. Thus the closure problem still very much remains.

Another problem with the eddy viscosity in the example above is that it fails to calculate the vertical components of the Reynolds stress and turbulent heat flux. An attempt at such a computation is shown in Figure 4.6 where the vertical turbulent heat flux is shown to be severely underestimated. Clearly the value of the eddy viscosity in the vertical direction must be different than in the radial direction. In other words, the turbulence for which a constitutive equation is being written is *not an isotropic “medium”*. In fact, in this specific example the problem is that the vertical component of the heat flux is produced more by the interaction of buoyancy and the turbulence, than it is by the working of turbulence against

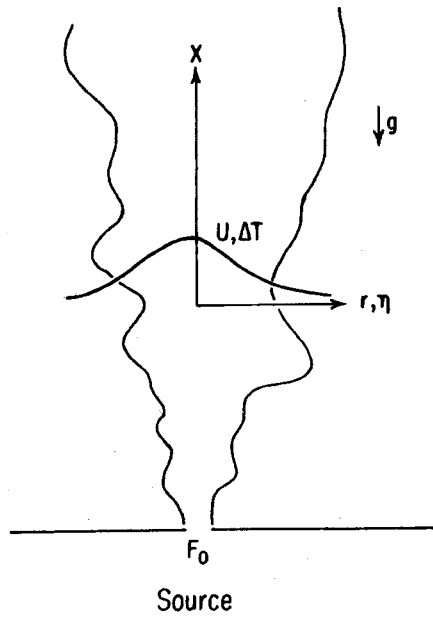


Figure 4.1: Schematic of axisymmetric plume

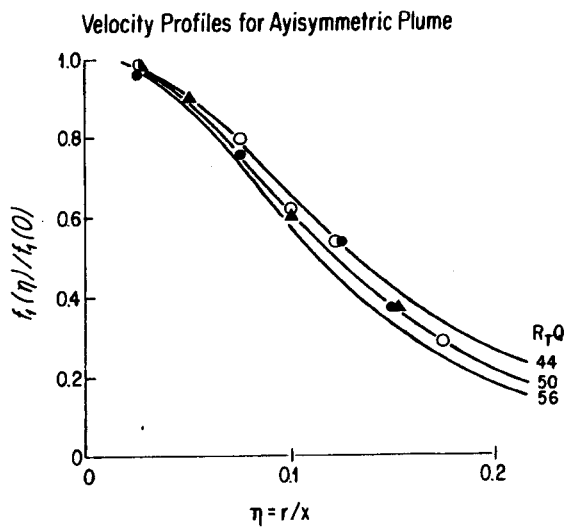


Figure 4.2: Mean velocity profiles for axisymmetric plume

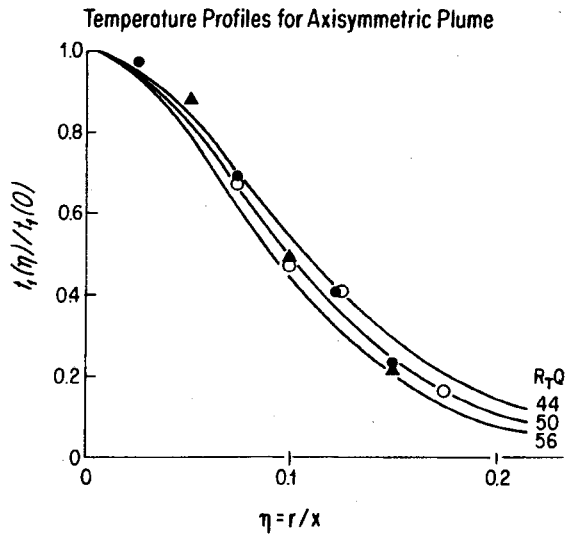


Figure 4.3: Mean temperature profiles for axisymmetric plume

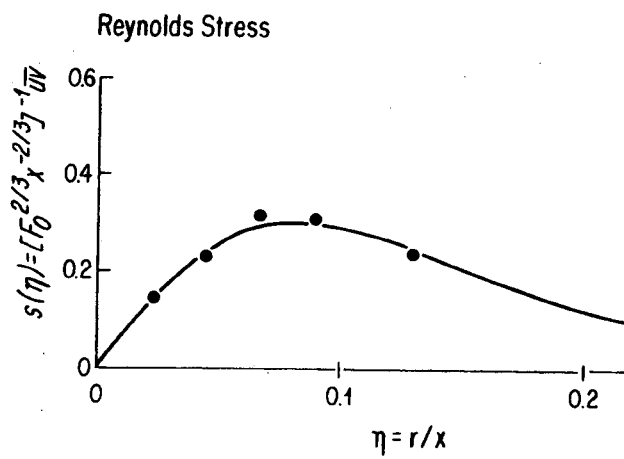


Figure 4.4: Reynolds shear stress profiles for axisymmetric plume

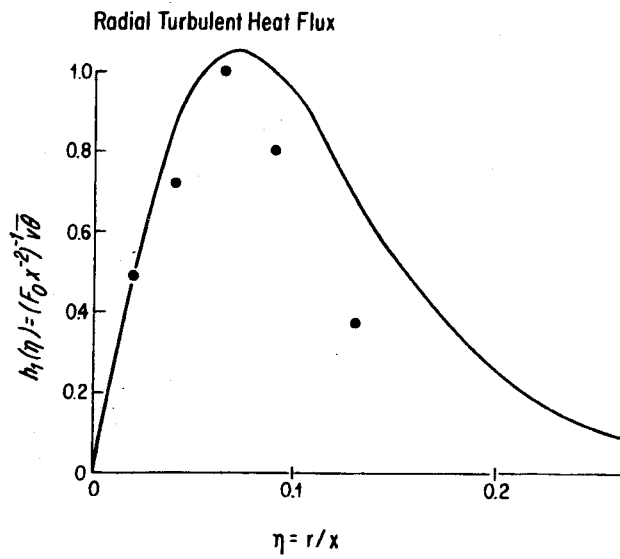


Figure 4.5: Radial turbulent heat flux for axisymmetric plume

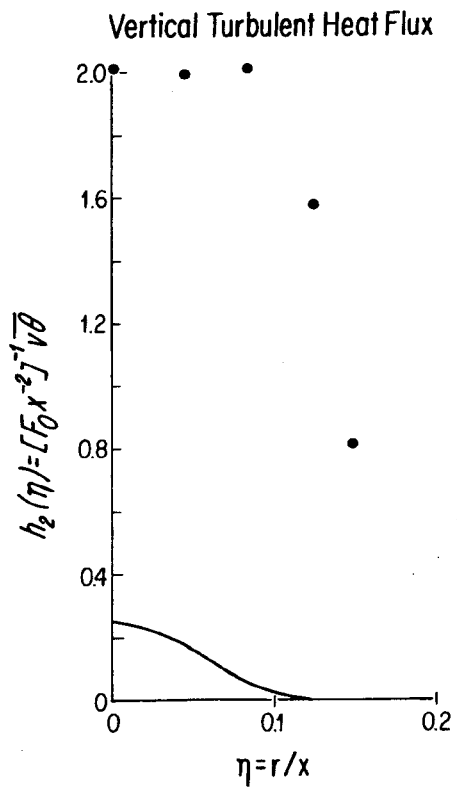


Figure 4.6: Vertical turbulent heat flux for axisymmetric plume

mean gradients in the flow. We will discuss this in more detail in the next chapter when we consider the turbulence energy balances, but note for now that simple gradient closure models never work unless gradient production dominates. This rules out many flows involving buoyancy, and also many involving recirculations or separation where the local turbulence is convected in from somewhere else.

A more general form of constitutive equation which would allow for the non-isotropic nature of the “medium” (in this case the turbulence itself) would be

$$-\rho\langle u_i u_j \rangle + \frac{1}{3}\langle u_k u_k \rangle \delta_{ij} = \mu_{ijkl} \left[S_{kl} - \frac{1}{3} S_{mm} \delta_{kl} \right] \quad (4.21)$$

This closure relation allows each component of the Reynolds stress to have its own unique value of the eddy viscosity. It is easy to see that it is unlikely this will solve the closure problem since the original six unknowns, the $\langle u_i u_j \rangle$, have been traded for eighty-one new ones, μ_{ijkl} . Even if some can be removed by symmetries, the remaining number is still formidable. More important than the number of unknowns, however, is that there is no independent or general means for selecting them without considering a particular flow. This is because *turbulence is indeed a property of the flow, not of the fluid.*

4.7 The Reynolds Stress Equations

It is clear from the preceding section that the simple idea of an eddy viscosity might not be the best way to approach the problem of relating the Reynolds stress to the mean motion. An alternative approach is to try to derive dynamical equations for the Reynolds stresses from the equations governing the fluctuations themselves. Such an approach recognizes that the Reynolds stress is really a functional⁵ of the velocity; that is, the stress at a point depends on the velocity everywhere and for all past times, not just at the point in question and at a particular instant in time.

The analysis begins with the equation for the instantaneous fluctuating velocity, equation 4.16. This can be rewritten for a Newtonian fluid with constant viscosity as:

$$\rho \left[\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} \right] = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ji}^{(v)}}{\partial x_j} - \rho \left[u_j \frac{\partial U_i}{\partial x_j} \right] - \rho \left\{ u_j \frac{\partial u_i}{\partial x_j} - \langle u_j \frac{\partial u_i}{\partial x_j} \rangle \right\} \quad (4.22)$$

Note that the free index in this equation is i . Also, since we are now talking about turbulence again, the capital letters represent mean or averaged quantities.

Multiplying equation 4.22 by u_k and averaging yields:

$$\begin{aligned} \rho \left[\langle u_k \frac{\partial u_i}{\partial t} \rangle + U_j \langle u_k \frac{\partial u_i}{\partial x_j} \rangle \right] = & - \langle u_k \frac{\partial p}{\partial x_i} \rangle + \langle u_k \frac{\partial \tau_{ji}^{(v)}}{\partial x_j} \rangle \\ & - \rho \left[\langle u_k u_j \rangle \frac{\partial U_i}{\partial x_j} \right] - \rho \left\{ \langle u_k u_j \frac{\partial u_i}{\partial x_j} \rangle \right\} \end{aligned} \quad (4.23)$$

⁵A functional is a function of a function

Now since both i and k are free indices they can be interchanged to yield a second equation given by⁶:

$$\begin{aligned} \rho \left[\langle u_i \frac{\partial u_k}{\partial t} \rangle + U_j \langle u_i \frac{\partial u_k}{\partial x_j} \rangle \right] = & - \langle u_i \frac{\partial p}{\partial x_k} \rangle + \langle u_i \frac{\partial \tau_{kj}^{(v)}}{\partial x_j} \rangle \\ & - \rho \left[\langle u_i u_j \rangle \frac{\partial U_k}{\partial x_j} \right] - \rho \left\{ \langle u_i u_j \frac{\partial u_k}{\partial x_j} \rangle \right\} \end{aligned} \quad (4.24)$$

Equations 4.23 and 4.24 can be added together to yield an equation for the Reynolds stress,

$$\begin{aligned} \frac{\partial \langle u_i u_k \rangle}{\partial t} + U_j \frac{\partial \langle u_i u_k \rangle}{\partial x_j} = & - \frac{1}{\rho} \left[\langle u_i \frac{\partial p}{\partial x_k} \rangle + \langle u_k \frac{\partial p}{\partial x_i} \rangle \right] \\ & - \left[\langle u_i u_j \frac{\partial u_k}{\partial x_j} \rangle + \langle u_k u_j \frac{\partial u_i}{\partial x_j} \rangle \right] \\ & + \frac{1}{\rho} \left[\langle u_i \frac{\partial \tau_{kj}^{(v)}}{\partial x_j} \rangle + \langle u_k \frac{\partial \tau_{ji}^{(v)}}{\partial x_j} \rangle \right] \\ & - \left[\langle u_i u_j \rangle \frac{\partial U_k}{\partial x_j} + \langle u_k u_j \rangle \frac{\partial U_i}{\partial x_j} \right] \end{aligned} \quad (4.25)$$

It is customary to rearrange the first term on the right hand side in the following way:

$$\begin{aligned} \left[\langle u_i \frac{\partial p}{\partial x_k} \rangle + \langle u_k \frac{\partial p}{\partial x_i} \rangle \right] = & \langle p \left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right] \rangle \\ & + \frac{\partial}{\partial x_j} [\langle p u_i \rangle \delta_{kj} + \langle p u_k \rangle \delta_{ij}] \end{aligned} \quad (4.26)$$

The first term on the right is generally referred to as the *pressure strain-rate* term. The second term is written as a divergence term, and is generally referred to as the *pressure diffusion* term. We shall see later that divergence terms can never create nor destroy anything; they can simply move it around from one place to another.

The third term on the right-hand side of equation 4.25 can similarly be rewritten as:

$$\begin{aligned} \left[\langle u_i \frac{\partial \tau_{kj}^{(v)}}{\partial x_j} \rangle + \langle u_k \frac{\partial \tau_{ji}^{(v)}}{\partial x_j} \rangle \right] = & - \left[\langle \tau_{ij}^{(v)} \frac{\partial u_k}{\partial x_j} \rangle + \langle \tau_{kj}^{(v)} \frac{\partial u_i}{\partial x_j} \rangle \right] \\ & + \frac{\partial}{\partial x_j} [\langle u_i \tau_{kj}^{(v)} \rangle + \langle u_k \tau_{ij}^{(v)} \rangle] \end{aligned} \quad (4.27)$$

⁶Alternatively equation 4.16 can be rewritten with free index k , then multiplied by u_i and averaged

The first of these is also a divergence term. For a Newtonian fluid, the last is the so-called “dissipation of Reynolds stress” by the turbulence viscous stresses. This is easily seen by substituting the Newtonian constitutive relation to obtain:

$$\frac{1}{\rho} \left[\langle \tau_{ij}^{(v)} \frac{\partial u_k}{\partial x_j} \rangle + \langle \tau_{kj}^{(v)} \frac{\partial u_i}{\partial x_j} \rangle \right] = 2\nu \left[\langle s_{ij} \frac{\partial u_k}{\partial x_j} \rangle + \langle s_{kj} \frac{\partial u_i}{\partial x_j} \rangle \right] \quad (4.28)$$

It is not at all obvious what this has to do with dissipation, but it will become clear later on when we consider the trace of the Reynolds stress equation, which is the *kinetic energy* equation for the turbulence.

Now if we use the same trick from before using the continuity equation, we can rewrite the second term on the right-hand side of equation 4.25 to obtain:

$$\left[\langle u_i u_j \frac{\partial u_k}{\partial x_j} \rangle + \langle u_k u_j \frac{\partial u_i}{\partial x_j} \rangle \right] = \frac{\partial}{\partial x_j} \langle u_i u_k u_j \rangle \quad (4.29)$$

This is also a divergence term.

We can use all of the pieces we have developed above to rewrite equation 4.25 as:

$$\begin{aligned}
\frac{\partial}{\partial t} \langle u_i u_k \rangle + U_j \frac{\partial}{\partial x_j} \langle u_i u_k \rangle &= \left\langle \frac{p}{\rho} \left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_i}{\partial x_k} \right] \right\rangle \\
+ \frac{\partial}{\partial x_j} \left\{ -\frac{1}{\rho} [\langle p u_k \rangle \delta_{ij} + \langle p u_i \rangle \delta_{kj}] - \langle u_i u_k u_j \rangle \right. \\
&\quad \left. + 2\nu [\langle s_{ij} u_k \rangle + \langle s_{kj} u_i \rangle] \right\} \\
- \left[\langle u_i u_j \rangle \frac{\partial U_k}{\partial x_j} + \langle u_k u_j \rangle \frac{\partial U_i}{\partial x_j} \right] \\
- 2\nu \left[\langle s_{ij} \frac{\partial u_k}{\partial x_j} \rangle + \langle s_{kj} \frac{\partial u_i}{\partial x_j} \rangle \right] & \quad (4.30)
\end{aligned}$$

This is the so-called **Reynolds Stress Equation** which has been the primary vehicle for much of the turbulence modeling efforts of the past few decades.

The left hand side of the Reynolds Stress Equation can easily be recognized as the rate of change of Reynolds stress following the mean motion. It seems to provide exactly what we need: nine new equations for the nine unknowns we cannot account for. The problems are all on the right-hand side. These terms are referred to respectively as

1. the pressure-strain rate term
2. the turbulence transport (or divergence) term
3. the “production” term, and
4. the “dissipation” term.

Obviously these equations do not involve only U_i and $\langle u_i u_j \rangle$, but depend on many more new unknowns.

It is clear that, contrary to our hopes, we have not derived a single equation relating the Reynolds stress to the mean motion. Instead, our Reynolds stress transport equation is exceedingly complex. Whereas the process of averaging the equation for the mean motion introduced only six new independent unknowns, the Reynolds stress, $\langle u_i u_j \rangle$, the search for a transport equation which will relate these to the mean motion has produced many more unknowns. They are:

$$\langle p u_i \rangle - 3 \text{ unknowns} \quad (4.31)$$

$$\langle u_i s_{jk} \rangle - 27 \quad (4.32)$$

$$\langle s_{ij} s_{jk} \rangle - 9 \quad (4.33)$$

$$\langle u_i u_k u_j \rangle - 27 \quad (4.34)$$

$$\left\langle p \frac{\partial u_i}{\partial x_j} \right\rangle - 9 \quad (4.35)$$

$$TOTAL - 75 \quad (4.36)$$

Not all of these are independent, since some can be derived from the others. Even so, our goal of reducing the number of unknowns has clearly not been met.

Equations governing each of these new quantities can be derived from the original dynamical equations, just as we did for the Reynolds stress. Unfortunately new quantities continue to be introduced with each new equation, and at a faster rate than the increase in the number of equations. Now the full implications of the closure problem introduced by the Reynolds decomposition and averaging has become apparent. No matter how many new equations are derived, the number of new unknown quantities introduced will always increase more rapidly.

4.8 Thermally varying flows

And analogous set of equations for the turbulent heat fluxes, $-\langle \theta u_j \rangle$, and the mean square temperature fluctuations, $\langle \theta^2 \rangle$, can also be derived. This section of these notes is incomplete at this submission. But we shall encounter them in the papers considered later.

4.9 Final comment

Our attempt to solve the turbulence problem by considering averages illustrates a general principle. Any time we try to fool Mother Nature by averaging out her details, she gets her revenge by leaving us with a closure problem — more equations than unknowns. In thermodynamics, we tried to simplify the consideration of molecules by averaging over them, and were left with the need for an equation of state. In heat transfer, we tried to simplify considerations by which molecules transfer their kinetic energy, and found we were lacking a relation between the heat flux and the temperature field. And in fluid mechanics, we tried to simplify consideration of the “mean” motion of molecules and ended up with viscous stress. In all of these cases we were able to make simple physical models which worked at least some of the time; e.g., ideal gas, Fourier-Newtonian fluid. And these models all worked because we were able to make assumptions about the underlying molecular processes and assume them to be independent of the macroscopic flows of interest. Unfortunately such assumptions are rarely satisfied in turbulence.

It should be obvious by now that the turbulence closure problem will not be solved by the straight-forward derivation of new equations, nor by direct analogy with viscous stresses. Rather, *closure attempts will have to depend on an intimate knowledge of the dynamics of the turbulence itself*. Only by understanding how the turbulence behaves can one hope to *guess* an appropriate set of constitutive equations **AND** *understand the limits of them*. This is, of course, another consequence of the fact that *the turbulence is a property of the flow itself, and not of the fluid!*

Chapter 5

Plumes and Thermals

This chapter is incomplete, but the focus will be on the canonical free shear flows of natural convection. These are particularly interesting since the averaged equations admit to similarity solutions. Similarity solutions are about the closest thing we have in turbulence to an exact solution which we can use to evaluate our experiments, numerical solutions and closure models.

We will consider a couple of the papers generated by me and my students over the years (mistakes and all). They are available on the website noted earlier:

www.turbulence-online.com

The papers of interest in this chapter are:

1. Shabbir, A. and W.K. George, (1994) Experiments on a Round Turbulent Buoyant Plume. *Journal of Fluid Mechanics*, 275, 1-32.
2. Baker, C.B., D.B. Taulbee and W.K. George (1982) An Analysis of the Buoyant Jet. *Heat Transfer 1982, Proc. of the 7th Inter. Heat Trans. Conf., Munich*, 131 - 144, Hemisphere Press, NY.
3. Beuther, P.D. and W.K. George (1982) Measurement of the Turbulent Energy and Temperature Balances in an Axisymmetric Buoyant Plume in a Stably Stratified Environment. *Heat Trans. 1982, Proc. of the 7th Inter. Heat Trans. Conf., Munich*, 163 - 178, Hemisphere Press, NY.

Chapter 6

Wall-bounded Flows

This chapter is incomplete. But the goal of this chapter is to discuss some of the unique problems of natural convection near to vertical and horizontal surfaces. The theoretical understanding of these flows is not in a very satisfactory state, and my own views have changed with time. There is even reason to suspect that the horizontal surface does not even have a statistically steady solution. The most important point (at least for this workshop) to be gleaned from all these papers is that in order for the ratio of length scales to be sufficiently large for the turbulence to matter, the Rayleigh or Grashof numbers must be huge.

The following papers will provide a useful basis for discussion. All are available from:

www.turbulence-online.com

1. George, W.K. and S.P. Capp (1979) An Analysis of Natural Convection Turbulent Boundary Layers Next to Heated Vertical Surfaces. *Int. J. of Heat and Mass Trans.*, 22, 813-826.
2. Wosnik, M. and W.K. George (1994) Another Look at the Turbulent Natural Convection Boundary Layer Next to Heated Vertical Surfaces. *Int. Symp. of Turb. Heat and Mass Transfer*, V1,14.5.1- 14.5.6.
3. Shiri, A. and W.K. George (2009) Turbulent natural convection in a differentially heated vertical channel. *Proceedings of 2008 ASME Heat Transfer Conference*, paper HT2008-56333.
4. George, W.K. (1989) A New Heat Transfer Law for Turbulent Natural Convection Between Heated Horizontal Surfaces of Large Extent, 26th ASME Nat. Heat Trans. Conf. and Exhib., Phila, PA, Aug. 6-9, ASME 1989 HTD, 107, 51-61.